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TRIGONOMETRY FOR BEGINNERS

The Y M C Co.

TRIGONOMETRY FOR BEGINNERS

BY THE

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REVISED AND ENLARGED

FOR THE USE OF AMERICAN SCHOOLS

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PREFACE

THE revision of the "Trigonometry for Beginners" differs from that of the original work, chiefly in the following particulars:

The subject matter of Chapter VII. formerly followed that of Chapters VIII. and IX.; the addition formulæ are proved for angles of any magnitude, and for more than two angles; a chapter on Inverse Trigonometric Functions and two chapters on Spherical Trigonometry have been added; logarithmic and trigonometric tables have been inserted. The rearrangement has necessitated minor changes in almost every chapter. Throughout the book, the question of ambiguity of solution has received careful attention. It is believed that the clear, simple presentation which characterized the original work has been retained.

It has been the endeavor to make definitions that need not be unlearned later; to give proofs, rigorous for the general plane angle; to present as much material as the student will master in a first course; and to present such material as will serve him best in his later studies. The proofs of many propositions are left as exercises for the student. These are formulated, and placed in the body of the text.

The lists of examples in the plane trigonometry are, for the most part, those of the original work. Some of the exercises in spherical trigonometry are selected from other texts.

Those desiring a shorter course may omit the chapters and the articles marked with an asterisk.

I acknowledge my indebtedness to Dr. Frank L. Sevenoak, who kindly permitted the use of his tables, and to friends who aided me by suggestions.

JOHN A. MILLER.

JUNE, 1896.

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TRIGONOMETRY FOR BEGINNERS

CHAPTER I

DEFINITIONS

1. The primary object of Trigonometry was, as its name implies, to measure (or solve) triangles; *i.e.* having given the measure of certain parts of a triangle, *e.g.* two sides and its included angle, to compute the remaining parts. In a broader and now universally accepted sense, Trigonometry embraces, in addition to the solution of triangles, all investigations of the relations existing among certain ratios intimately associated with an angle. These ratios are defined in Art. 26.

This branch of the subject is sometimes called **Angular Analysis.**

2. The **figures** with which we shall be concerned in our study of Trigonometry are, with the exception of the *line* and the *angle*, the same as those of Geometry; *i.e.* they are subject to the same limitations and possess the same properties as those of Geometry. For example, the sum of the interior angles of a triangle equals two right angles in Trigonometry as well as in Geometry.

3. The **line** of Trigonometry differs from the line of Geometry, in that, in Trigonometry, it is sometimes of advantage to distinguish between lines drawn in *opposite* directions. [See Art. 47.] For the present, however, we shall not make this distinction.

4. By the **angle** XOP (Fig. 1, Fig. 2, Fig. 3), in Trigonometry, is not meant the *present inclination* of the lines OX and OP as in Geometry, but the amount of turning which OP has done about the point O, in coming from its *initial* position OX, to its *final* position OP.

ILLUSTRATION. Suppose a race to be run around a circular course. The position of any one of the competitors would be known, if we know that he has described a certain angle about the centre of the course. Thus, if the distance to be run is three times around, the line joining each competitor to the centre would have to describe an angle of 12 right angles.

When we say that a competitor has described an angle of $6\frac{1}{2}$ right angles, we give not only his present position, but the total distance he has gone. He would, in such a case, have gone a little more than one and a half times around the course.

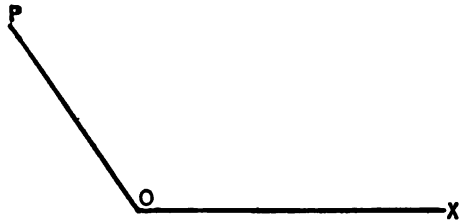


FIG. 1.

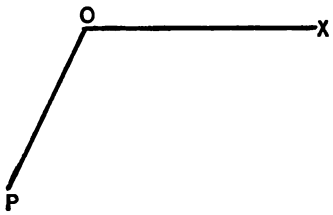


FIG. 2.

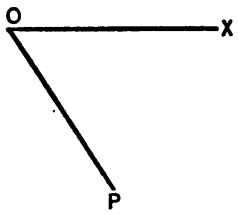


FIG. 3.

It is evident from this definition that a trigonometrical angle may have any magnitude however great. It is well to notice that angle XOP is the amount of turning that *has been done*. In other words, it is the *result* of the turning, not the *process*.

5. The geometrical **representation** of a trigonometric angle depends only on the initial and final positions of the line OP. Hence the figure XOP (Fig. 1, Art. 4) may be the geometrical representative of an unlimited number of trigonometrical angles.

(i.) The angle XOP may represent the angle less than two right angles, as in Geometry.

In this case, OP has turned from the position OX into the position OP by turning about O in the direction *contrary* to that of the hands of a watch.

(ii.) The angle XOP may represent the angle described by OP in turning from the position OX into the position OP in the *same* direction as that of the hands of a watch.

In the first case it is usual to say that the angle XOP is a *positive* angle; in the second case it is a *negative* angle.

(iii.) The angle XOP may be the geometrical representation of any of the trigonometrical angles formed by any number of complete revolutions in the *positive* or in the *negative* direction, added to either of the first two angles mentioned in (i.) and (ii.).

We may express (iii.) thus: XOP is the geometrical representative of $XOP + 4n$ right angles, where n is any integer.

6. Definitions. O is called the **origin.** The line OX is the **initial line.** The line OP is the **revolving line or radius vector.** When referring to the angle XOP the lines OX and OP are called the **sides** of the angle, and O is called its **vertex.**

7. To add angle $X'OP'$ to angle XOP, both being positive, revolve OP from its final position when it represents angle XOP,

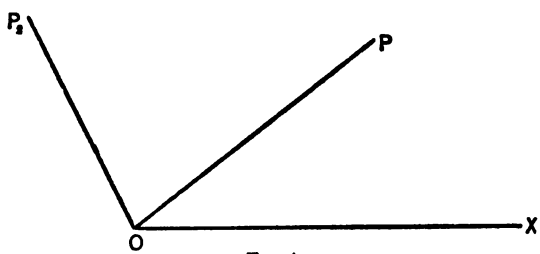


Fig. 4.

through an angle equal to angle $X'OP'$; call this position OP_2. Then,

$$\angle XOP_2 = \angle XOP + \angle X'OP'.$$

EXAMPLES. I.

Give a geometrical representation of each of the following angles, the initial line being drawn in each case from the origin towards the right:

1. $+3$ right angles.
2. $+5$ right angles.
3. $+4\frac{1}{2}$ right angles.
4. $+7\frac{1}{4}$ right angles.
5. -1 right angle.
6. $10\frac{2}{3}$ right angles.
7. $-10\frac{1}{3}$ right angles.
8. $+4$ right angles.
9. -4 right angles.
10. $4n$ right angles.
11. $(4n+2)$ right angles.
12. $-(4n+\frac{1}{2})$ right angles.
13. $1\frac{1}{2}$ right angles $+ 2\frac{1}{2}$ right angles.
14. $3\frac{1}{2}$ right angles $- \frac{1}{2}$ right angle.

8. Certain propositions which the student has proved while studying plane Geometry will be referred to very frequently, and quoted without proof.

The principal ones are:

a. The Pythagorean theorem.

b. Conditions under which two triangles are similar.

c. Homologous sides of similar triangles are proportional, and homologous angles are equal.

d. I. The ratio $\frac{\text{circumference of a circle}}{\text{diameter of a circle}}$ is a certain fixed number.

II. It is an *incommensurable* number.

III. It is 3.14159265+ ···.

9. When we say that this number is incommensurable, we mean that its exact value cannot be stated as an *arithmetical* fraction.

It also happens that we have no short *algebraical* expression such as a surd, or combination of surds, which represents it exactly, so that we have no *numerical* expression whatever, arithmetical nor algebraical, to represent *exactly* the ratio of the circumference of a circle to its diameter.

Hence the universal custom has arisen, of denoting its *exact* value by the letter π.

Thus π stands *always* for **the exact value of a certain incommensurable number,** whose approximate value is 3.14159265, and which **is the ratio of the circumference of any circle to its diameter.**

It cannot be too carefully impressed on the student's memory that π stands for this number 3.14159265 ···, etc., and for nothing else; just as 180 stands for the number one hundred and eighty, and for nothing else.

We may notice that $\frac{22}{7} = 3.\dot{1}4285\dot{7}$.

So that $\frac{22}{7}$ and π differ by less than a thousandth part of their value.

CHAPTER II

MEASUREMENT OF ANGLES

10. It is usual to say that we have **measured** any concrete quantity, when we have found **how many** times it contains some familiar quantity of the same kind.

We say, for example, that we have measured a line, when we have found *how many* feet it contains. We say that we have measured a field, when we have found out *how many* acres or how many square yards it contains.

To know the measurement of any quantity, then, we must have two things. First, we must have a *unit*, or standard of reference, of the *same kind* as the thing measured. Secondly, we must have the *measure*, or the *number of times* the thing measured contains the unit, or standard quantity.

Hence, the **measure** of a quantity is the **number**, and the **unit** is the **concrete quantity**, by means of which it is measured.

EXAMPLE 1. *A line contains* 261 feet; that is, 261 times a foot. Here the *measure* or number is 261, and the *unit* a foot.

EXAMPLES. II.

1. What is the measure of 1 mile when a chain of 66 feet is the unit?

2. What is the measure of an acre when a square whose side is 22 yards is the unit?

3. The length of an Atlantic cable is 2300 miles and the length of the cable from England to France is 21 miles. Express the length of the first in terms of the second as a unit.

4. The measure of a certain field is 22 and the unit 1100 square yards: express the area of the field in acres.

5. Find the measure of a miles when b yards is the unit.

6. The measure of a certain distance is a when the unit is c feet. Express the distance in yards.

11. Measurement of angles. There are two common methods of measuring angles.

(i.) The rectangular measure.
(ii.) The circular measure.

Rectangular Measure

12. Angles are always measured *in practice* with the **right angle** (or part of the right angle) as unit.

The reasons why the right angle is chosen for a unit are:

(i.) All right angles are equal to one another.

(ii.) A right angle is practically easy to draw.

(iii.) It is an angle whose size is very familiar.

13. The right angle is divided into 90 equal parts, each of which is called a **degree**; each degree is subdivided into 60 equal parts, each of which is called a **minute**; and each minute is again subdivided into 60 equal parts, each of which is called a **second.**

Instruments used for measuring angles are subdivided accordingly; and the size of an angle is known when, with such an instrument, it has been observed that the angle contains a certain number of degrees, and a certain number of minutes beyond the number of complete degrees, and a certain number of seconds beyond the number of complete minutes.

Thus an angle might be recorded as containing 79 degrees + 18 minutes + 36.4 seconds.

Degrees, minutes, and seconds are indicated respectively by the symbols °, ′, ″, and the above angle would be written 79° 18′ 36.4″.

14. An angle given in degrees, minutes, and seconds may be expressed as the decimal of a right angle by the usual method.

Example. Express 39° 4′ 27″ as the decimal of a right angle.

60	27 seconds
60	4.45 minutes
90	39.7416666 etc. degrees

.441574074 etc. right angles

.441574074 of a right angle, *Ans.*

Note. The French proposed to call the 100th part of a right angle a grade (written 3^g), the 100th part of a grade a minute (written $3^{\backslash}$), the 100th part of a minute a second (written $3^{\backslash\backslash}$). So that 1.437275 right angles would be read $143^g\ 72^{\backslash}\ 75^{\backslash\backslash}$. The decimal method of subdividing the right angles has never been used.

Circular Measure

15. Definition. A **radian** is an angle at the centre of a circle, subtended by an arc equal in length to the radius of the circle. Thus if in the circle RPS, whose centre is O, arc RP = radius OR, then, angle ROP is a radian.

16. We shall now prove that the radian is a constant angle; or stating the same thing differently, we are about to prove that if we take any number of different circles, and measure on the circumference of each an arc equal in length to its radius, then the angles at the centres of these circles which stand on these arcs respectively, will be all of the same size.

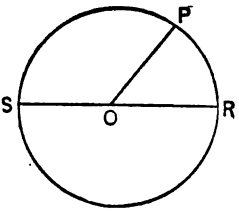


FIG. 5.

17. *To prove that all radians are equal to one another.*

Since the radian at the centre of a circle stands on an arc equal in length to the radius,

and an angle of two right angles at the centre of a circle stands on half the circumference,

and since angles at the centre of a circle are to one another as the arcs on which they stand (Geom.),

then, $$\frac{\text{a radian}}{\text{2 right angles}} = \frac{\text{radius}}{\text{semi-circumference}}$$
$$= \frac{\text{diameter}}{\text{circumference}} = \frac{1}{\pi}.$$

Therefore a radian $= \frac{1}{\pi}$ of 2 right angles,

$=$ a certain fixed fraction of 180°. (Art. 8.)

Thus the radian possesses the qualification most essential in a unit; viz. it is always the same.

18. The reasons why a radian is used as a unit are:

(i.) All radians are equal to one another.

(ii.) Its use simplifies many formulæ in Theoretical Trigonometry.

19. Since $$1 \text{ radian} = \frac{2 \text{ rt. } \angle\text{s}}{\pi},$$ (Art. 17.)

$$\therefore \quad 1 \text{ radian} = 57.2957°.$$

20. Since $$1 \text{ radian} = \frac{2 \text{ rt. } \angle\text{s}}{\pi},$$

$$\therefore \quad \pi \text{ radians} = 2 \text{ rt. } \angle\text{s} = 180°,$$

$$\frac{\pi}{2} \text{ radians} = 1 \text{ rt. } \angle = 90°,$$

$$\frac{3\pi}{2} \text{ radians} = 3 \text{ rt. } \angle\text{s} = 270°,$$

and so on.

21. A "c" is used to indicate radians, just as "°" is used to indicate degrees. Thus θ^c is read "θ radians," just as $A°$ is read "A degrees." When the measure of an angle is expressed as some *multiple* of π, *e.g.* $\frac{\pi}{2}$, *the unit being the radian,* common usage has sanctioned the omission of the "c" after the measure; thus, $\frac{\pi^c}{2}$ is usually written $\frac{\pi}{2}$. This practice sometimes confuses the beginner. Yet, with a little care no confusion need arise. We need only remember that π is a mere number (Art. 8); that in the expression "the angle π" or kindred expressions *some unit must* be implied, and that by common agreement this unit is the radian, so that π is often used for 180°, meaning, then, π radians (Art. 20). This does not exclude the angle $\pi°$; but if a degree is the unit, it is so expressed.

Another agreement sometimes made, is that the Greek letters α, β, etc., are used to express the measure of angles when the radian is the unit of measure; while, the Roman letters A, B, C, etc., are used when the unit of measure is the degree. The distinction, however, is not universally observed.

22. *To find the number of degrees in an angle containing a given number of radians.*

Let $D =$ number of degrees in the angle.
Let $\alpha =$ number of radians in the angle.

$$1^c = \left(\frac{180}{\pi}\right)^\circ. \qquad \text{(Art. 17.)}$$

$$\therefore\ \alpha^c = \left(\frac{\alpha \times 180}{\pi}\right)^\circ.$$

$$\therefore\ \frac{\alpha \times 180}{\pi} = D, \text{ the } \textit{number} \text{ of degrees} \qquad (1)$$

which was to be found.

We may in a similar manner *find the number of radians in an angle containing a given number of degrees.* Or, simply solving (1) for α, we have $\alpha = \frac{\pi D}{180°}$, the number of radians in an angle containing D degrees.

The problem may be thus solved: $\frac{D°}{180°} = \frac{\alpha^c}{\pi^c}$, for each fraction is the *ratio* of the same angle to two right angles.

EXAMPLE. Find number of degrees in two radians. We have

$$\frac{D}{180}=\frac{2}{\pi}\ \therefore\ D=\frac{360}{\pi}.$$

23. EXAMPLES. III.

I. Express each of the following angles as the decimal of a right angle:

1. $8° 15' 27''$.
2. $6° 4' 30''$.
3. $97° 5' 15''$.
4. $16° 14' 19''$.
5. $132° 6'$.
6. $49°$.

Express in degrees, minutes, and seconds:

7. .01375 right angles.
8. .0875 right angles.
9. 1.704535 right angles.
10. .240025 right angles.
11. .180115 right angles.
12. .35 right angles.

II. Express the following angles in rectangular measure:

1. π.
2. $\frac{3\pi}{4}$.
3. 1^c.
4. 3^c.
5. 3.14159265^c, etc.
6. $\frac{2^c}{\pi}$
7. θ.
8. $.00814159^c$, etc.
9. 10π.

III. Express the following angles in circular measure:

1. $180°$.
2. $360°$.
3. $60°$.
4. $22\frac{1}{2}°$.
5. $1°$.
6. $57.295°$, etc.
7. $n°$.
8. $\frac{90°}{\pi}$.
9. A.

IV. Give a geometrical representation of the following trigonometrical angles. Draw the initial line from O, horizontally, to the right.

1. $\frac{\pi}{2}$.
2. 8π.
3. 5π.
4. $(2n+1)\pi$.
5. $(2n+\frac{1}{2})\pi$.

V. Find the ratio of

1. $45°$ to $\frac{3\pi}{4}$.
2. $1°$ to 1^c.
3. 1.75^c to $\frac{100°}{\pi}$.

24. Since angles at the centre of a circle are to one another as the arcs on which they stand (Geom.), therefore (see Fig. 6)

$$\frac{\text{an angle } ROP}{\text{one radian}}=\frac{\text{arc } RP}{\text{arc } RS}=\frac{\text{arc } RP}{\text{the radius}}.$$

Hence the angle $ROP=\frac{\text{arc } RP}{\text{the radius}}$ radians.

So that the **circular measure** of an angle (at the centre of a circle) is the **ratio of its arc to the radius.**

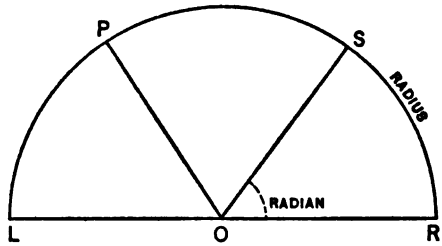


FIG. 6.

EXAMPLE. Find the number of degrees in the angle subtended by an arc 46 ft. 9 in. long, at the centre of a circle whose radius is 25 feet.

The angle stands on an arc of $46\frac{3}{4}$ feet, and the radian, at the centre of the same circle, stands on an arc of 25 feet.

$$\therefore \text{ the angle} = \frac{46\frac{3}{4}}{25} \text{ radians}, = \frac{187}{100} \times \frac{2 \text{ right angles}}{\pi},$$

$$= \frac{187}{100} \times \frac{180^\circ}{\pi} = 105.8^\circ \text{ nearly.}$$

25. EXAMPLES. IV.

(IN THE ANSWERS $\frac{22}{7}$ IS USED FOR π.)

1. Find the number of radians in an angle at the centre of a circle of radius 25 feet, which stands on an arc of $37\frac{1}{2}$ feet.

2. Find the number of degrees in an angle at the centre of a circle of radius 10 feet, which stands on an arc of 5π feet.

3. Find the number of right angles in the angle at the centre of a circle of radius $3\frac{2}{11}$ inches, which stands on an arc of 2 feet.

4. Find the length of the arc subtending an angle of $4\frac{1}{2}$ radians at the centre of a circle whose radius is 25 feet.

5. Find the length of an arc of 80 degrees on a circle of 4 feet radius.

6. The angle subtended by the diameter of the sun at the eye of an observer is $32'$; find approximately the diameter of the sun if its distance from the observer be 90,000,000 miles.

7. A railway train is travelling on a curve of half a mile radius at the rate of 20 miles an hour; through what angle has it turned in 10 seconds?

8. A railway train is travelling on a curve of two-thirds of a mile radius, at the rate of 60 miles an hour; through what angle has it turned in a quarter of a minute?

9. Find approximately the number of seconds contained in the angle which subtends an arc one mile in length at the centre of a circle whose radius is 4000 miles.

10. If the radius of a circle be 4000 miles, find the length of an arc which subtends an angle of $1''$ at the centre of the circle.

11. If in a circle whose radius is 12 ft. 6 in. an arc whose length is .6545 of a foot subtends an angle of 3 degrees, what is the ratio of the diameter of a circle to its circumference?

12. If an arc 1.309 feet long subtend an angle of $7\frac{1}{2}$ degrees at the centre of a circle whose radius is 10 feet, find the ratio of the circumference of a circle to its diameter.

13. On a circle 80 feet in radius it was found that an angle of $22^\circ\ 30'$ at the centre was subtended by an arc 31 ft. 5 in. in length; hence calculate to four decimal places the numerical value of the ratio of the circumference of a circle to its diameter.

14. If the diameter of the moon subtend an angle of $30'$, at the eye of an observer, and the diameter of the sun an angle of $32'$, and if the distance of the sun be 375 times the distance of the moon, find the ratio of the diameter of the sun to that of the moon.

15. Find the number of radians (*i.e.* the circular measure of) in $10''$ correct to 3 significant figures. (Use $\frac{355}{113}$ for π.)

16. Find the radius of a globe such that the distance measured upon its surface between two places in the same meridian, whose latitudes differ by $1^\circ\ 10'$, may be one inch.

17. By construction prove that the unit of circular measure is less than 60°.

18. On the 31st December the sun subtends an angle of $32'\ 36''$, and on 1st July an angle of $31'\ 32''$; find the ratio of the distances of the sun from the observer on those two days.

19. Show that the measure of the angle at the centre of a circle of radius r, which stands on an arc a, is $\frac{k \cdot a}{r}$, where k depends solely on the unit of angle employed.

Find k when the unit is (i.) a radian, (ii.) a degree.

20. The difference of two angles is $\frac{1}{3}\pi$, and their sum 56°; find them.

21. Find the number of radians in an angle of n'.

22. Express in right angles and in radians the angles

(i.) of a regular hexagon,
(ii.) of a regular octagon,
(iii.) of a regular quindecagon.

23. Taking for unit the angle between the side of a regular quindecagon and the next side produced, find the measures (i.) of a right angle, (ii.) of a radian.

24. Find the unit when the sum of the measures of a degree and of the hundredth part of a right angle is 1.

25. What is the unit when the sum of the measures of a right angles and of b degrees is c?

26. The three angles of a triangle have the same measure when the units are $\frac{1}{90}$ of a right angle, $\frac{1}{100}$ of a right angle, and a radian respectively; find the measure.

27. The interior angles of an irregular polygon are in A.P.; the least angle is 120°; the common difference is 5°; find the number of sides.

CHAPTER III

THE TRIGONOMETRICAL RATIOS

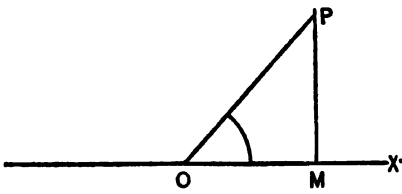


FIG. 7.

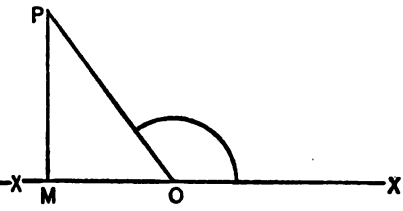


FIG. 8.

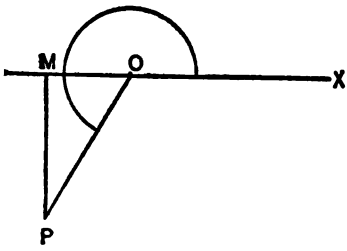


FIG. 9.

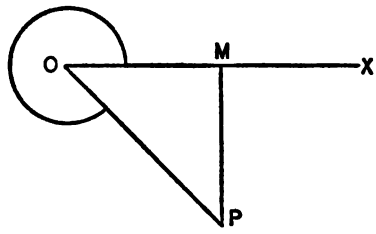


FIG. 10.

26. Given any angle, XOP, as in the figures. From P, any point in OP, the revolving line, a perpendicular PM is let fall on the initial line OX, as in Figs. 6 and 9, or on the initial line produced, as in Figs. 7 and 8. Six ratios may now be written, viz.,

27. I. $\frac{MP}{OP}$, II. $\frac{OM}{OP}$, III. $\frac{MP}{OM}$, IV. $\frac{OP}{MP}$, V. $\frac{OP}{OM}$, VI. $\frac{OM}{MP}$.

IV., V., VI. are respectively the reciprocals of I., II., and III.

28. These ratios are of much importance in all mathematical investigations. For convenience, they have been given the following names. If XOP be the angle A, then

I., $\frac{MP}{OP}$ is called the **sine of angle** A, and is read, sine A;

II., $\frac{OM}{OP}$ is called the **cosine of angle** A, and is read, cosine A;

III., $\frac{MP}{OM}$ is called the **tangent of angle** A, and is read, tangent A;

IV., $\frac{OP}{MP}$ is called the **cosecant of angle** A, and is read, cosecant A;

V., $\frac{OP}{OM}$ is called the **secant of angle** A, and is read, secant A;

VI. $\frac{OM}{MP}$ is called the **cotangent of** A, and is read, cotangent A.

29. The expressions *sine A, cosine A, tangent A, cosecant A, secant A, cotangent A* are abbreviated into *sin A, cos A, tan A, csc A* or *cosec A, sec A, cot A,* respectively.

The powers of the trigonometric ratios are expressed as follows:

$$(\sin A)^2 = \sin A \times \sin A, \text{ is written } \sin^2 A;$$
$$(\cos A)^3 = \cos A \cdot \cos A \cdot \cos A, \text{ is written } \cos^3 A;$$
$$(\tan A)^n \text{ is written } \tan^n A,$$

and so on.

The student must notice that "$\sin A$" is a *single symbol.* It is the *name* of a *number*, or *fraction,* belonging to the angle A; and if it be at any time convenient, we may denote *sin A* by a *single letter*, such as s or x. Also $\sin^2 A$ is an abbreviation for $(\sin A)^2$, that is, for $(\sin A) \times (\sin A)$. Such abbreviations are used because they are *convenient.*

The student who succeeds in the study of trigonometry must commit the preceding definitions to memory.

CHAPTER IV

THE ACUTE ANGLE

30. Thus far we have placed no limitations on the magnitude of the angle under consideration at any time. In the present chapter we shall confine our attention to angles lying *between* 0° *and* 90°. We shall, in Chapter VII., return to the consideration of the general angle.

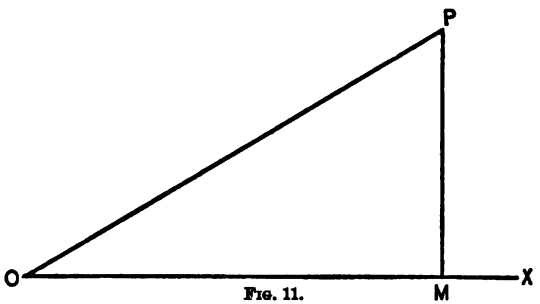


FIG. 11.

31. Given a right triangle OMP with the right angle at M. Then MOP is acute, as is also OPM. Let us consider angle MOP; call $\angle MOP$, A; then,

$$\sin A = \frac{MP}{OP} = \frac{\text{side of triangle opposite angle under consideration}}{\text{hypotenuse}},$$

or, briefly, $$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}};$$

$$\cos A = \frac{OM}{OP}$$

$$= \frac{\text{side of triangle adjacent to the angle under consideration}}{\text{hypotenuse}},$$

or, briefly, $$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}};$$

$$\tan A = \frac{MP}{OM} = \frac{\text{opposite side}}{\text{adjacent side}};$$

and similar expressions for the other ratios.

EXERCISE. Write the trigonometrical ratios of angle P.

32. Assuming that the angle XOP is less than 90°, we shall show

I. *That so long as*[1] *the angle remains unchanged, the ratios remain unchanged.*

II. *That a small change in the angle produces a change in each of the ratios.*

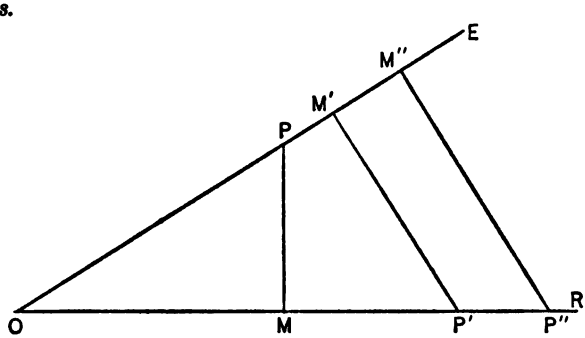


FIG. 12.

I. Take any angle ROE; let P be any point in OE, one of the lines containing the angle, and let P', P'' be any two points in OR, the other line containing the angle. Draw PM perpendicular to OR, and $P'M'$, $P''M''$ perpendiculars to OE.

Then the three triangles OMP, $OM'P'$, $OM''P''$ each contain a right angle, and they have the angle at O common; therefore their third angles must be equal.

Thus the three triangles are equiangular.

Therefore the ratios $\frac{MP}{OP}$, $\frac{M'P'}{OP'}$, $\frac{M''P''}{OP''}$ are all equal. (Geom.)

But each of these ratios is $\frac{\text{opposite side}}{\text{hypotenuse}}$ *with reference to the angle at* O; that is, they are each sin ROE.

Thus, sin ROE is the same *whatever* be the position of the point P on *either* of the lines containing the angle ROE.

Therefore sin ROE is always the same.

II. Let XOP and XOP' be two angles nearly equal. (See Fig. 13).

[1] We shall show (Art. 63) that this change must be, in general, less than 90°. However, our proof is rigorous for the proposition stated.

Let $OP = OP'$. (See I.)

Let $P'Q'$ and PQ be perpendicular to OX.

Since $\angle XOP'$[1] $\neq \angle XOP$; $P'Q' \neq PQ$ (Geometry);

$\therefore M'P' \neq MP$ (Geometry); $\therefore OM' \neq OM$ (Geometry);

$$\sin XOP' = \frac{OP'}{M'P'}; \qquad \sin XOP = \frac{MP}{OP}.$$

But,
$$\frac{M'P'}{OP'} \neq \frac{MP}{OP}.$$

$$\therefore \sin XOP' \neq \sin XOP.$$

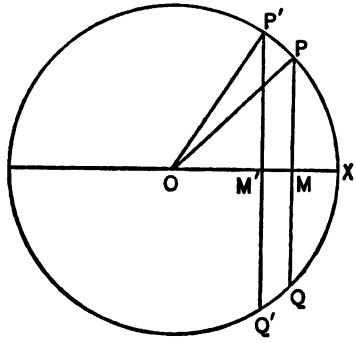


FIG. 13.

EXERCISE I. Prove Proposition I. of this article for the remaining ratios.

EXERCISE II. Prove Proposition II. of this article for the remaining ratios.

33. The student should observe carefully

(i.) that each *ratio*, such as $\frac{\text{opposite side}}{\text{hypotenuse}}$, is a mere *number;*

(ii.) that we have proved, in Art. 32, these ratios remain unchanged as long as the angle remains unchanged;

(iii.) that if the angle be altered ever so slightly, there is a consequent alternation in the value of these ratios;

(iv.) that since these ratios are all numbers, they are therefore algebraic quantities, and hence obey all the laws of ordinary algebra.

(v.) that there is a right angle in the same triangle as the angle referred to.

EXAMPLE. In Fig. 14, in which BDA is a right angle, find $\sin DBA$ and $\cos DBA$.

In this case

(i.) DBA is the *angle.*

(ii.) BDA is a *right angle* in the same triangle as the angle DBA.

(iii.) DA is the *side* opposite DBA and is perpendicular to BD.

(iv.) BA is the *hypotenuse.*

(v.) BD is the *adjacent side.*

[1] We shall use the sign $\neq$, $\not<$, $\not>$, to mean "is different from," "is not less than," "is not greater than," respectively.

Therefore $\sin DBA$, which is $\frac{\text{opposite side}}{\text{hypotenuse}}$, $= \frac{DA}{BA}$;

$\cos DBA$, which is $\frac{\text{adjacent side}}{\text{hypotenuse}}$, $= \frac{BD}{BA}$.

34. **EXAMPLES. V.**

1. In the triangle ABC, C being a right angle, $AB = 25$, $CB = 16$; find $\sin A$, $\cos A$.

2. If in the triangle ABC, C being a right angle, $AC = 2$, $BC = 4$, find $\sin B$, $\cos B$, and $\cot B$.

3. Let ACB be any angle, and let ABC and BDC be right angles (see Fig. 14). Write down *two* values for each of the following ratios: (i.) $\sin ACB$, (ii.) $\cos ACB$, (iii.) $\tan ACB$, (iv.) $\sin BAC$, (v.) $\cos BAC$, (vi.) $\tan BAC$.

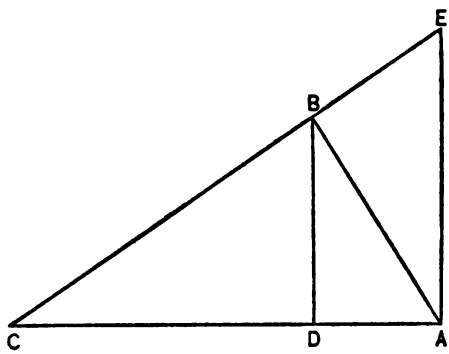


Fig. 14.

4. In Fig. 14 BDC, CBA, and EAC are right angles.

Write down (i.) $\sin DBA$, (ii.) $\sin BEA$, (iii.) $\sin CBD$, (iv.) $\cos BAE$, (v.) $\cos BAD$, (vi.) $\cos CBD$, (vii.) $\tan BCD$, (viii.) $\tan DBA$, (ix.) $\tan BEA$, (x.) $\tan CBD$, (xi.) $\sin DAB$, (xii.) $\sin BAE$.

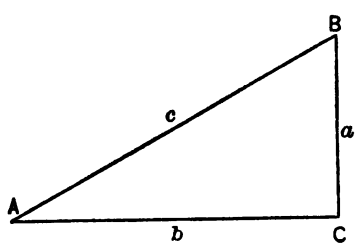


Fig. 15.

5. If ABC be any right-angled triangle with a right angle at C, and we let A, B, and C stand for the angles at A, B, and C respectively, and let a, b, and c be the measures of the sides opposite the angles A, B, and C respectively:

Show that $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$, $\tan A = \frac{a}{b}$.

Show also that $\sin^2 A + \cos^2 A = 1$.

Show also that (i.) $a = c \cdot \sin A$, (ii.) $b = c \cdot \sin B$, (iii.) $a = c \cdot \cos B$, (iv.) $b = c \cdot \cos A$, (v.) $\sin A = \cos B$, (vi.) $\cos A = \sin B$, (vii.) $\tan A = \cot B$.

6. The sides of a right-angled triangle are in the ratio 5 : 12 : 13; find the sine, cosine, and tangent of each acute angle of the triangle.

7. The sides of a right-angled triangle are in the ratio $1 : 2 : \sqrt{3}$; find the sine, cosine, and tangent of each acute angle of the triangle.

8. Prove that if A be either of the angles of the above two triangles, $\sin^2 A + \cos^2 A = 1$.

9. ABC is a right-angled triangle, C being the right angle. AB is 2 feet and AC is 1 foot; find the length of BC, and thence find the value of $\sin A$, $\cos A$, and $\tan A$.

10. ABC is a right-angled triangle, C being the right angle, $AB = \sqrt{2}$ ft. and $AC = 1$ foot; prove that $\sin A = \cos A = \sin B = \cos B$.

11. ABC is a right-angled triangle, C being the right angle; $BC = 1$ foot, and $AB = \sqrt{3}$ feet; find AC and $\sin A$ and $\sin B$.

CHAPTER V

ON THE TRIGONOMETRICAL RATIOS OF CERTAIN ANGLES

35. The trigonometrical ratios of an angle are *numerical quantities simply,* as their name ratio implies. They are in nearly all cases incommensurable numbers.

Their practical value has been found for all angles between 0° and 90°, which differ by 1′; and a list of these values will be found in any volume of Mathematical Tables.

The general method of finding trigonometrical ratios belongs to a more advanced part of the subject than the present, but there are certain angles whose ratios can be found in a simple manner.

36. *To find the* **sine, cosine,** *and* **tangent** *of an angle of* **45°.**

When one angle of a right-angled triangle is 45°, that is, the half of a right angle, the third angle must also be 45°. Hence 45° is one angle of an *isosceles* right-angled triangle.

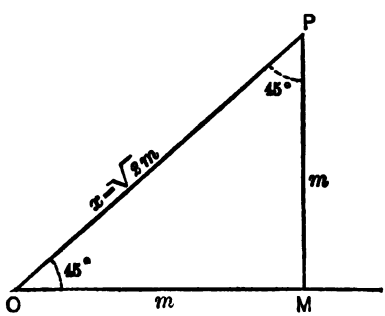


FIG. 16.

Let POM be an isosceles triangle such that PMO is a right angle, and $OM = MP$. Then $POM = OPM = 45°$.

Let the measures of OM and of MP each be m. Let the measure of OP be x.

Then $$x^2 = m^2 + m^2 = 2m^2.$$

$$\therefore\ x = \sqrt{2} \cdot m.$$

Hence, $$\sin 45^\circ = \sin POM = \frac{MP}{OP} = \frac{m}{\sqrt{2} \cdot m} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

$$\cos 45^\circ = \cos POM = \frac{OM}{OP} = \frac{m}{\sqrt{2} \cdot m} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

$$\tan 45^\circ = \tan POM = \frac{MP}{OM} = \frac{m}{m} = \frac{1}{1} = 1.$$

37. *To find the* **sine, cosine,** *and* **tangent** *of an angle of* **60°.**

Each angle in an *equilateral* triangle is 60°, because they are each one-third of 180°. If we draw a perpendicular from one of the angular points of the triangle to the side opposite, we get a right-angled triangle in which one angle is 60°.

Let OPQ be an equilateral triangle. Draw PM perpendicular to OQ. Then OQ is bisected in M.

Let the measure of OM be m; then that of OQ is $2m$, and therefore that of OP is $2m$.

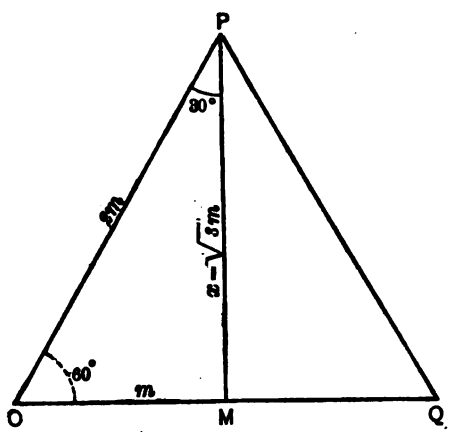


FIG. 17.

Let the measure of MP be x.

Then $$x^2 = (2m)^2 - m^2 = 4m^2 - m^2 = 3m^2.$$

$$\therefore\ x = \sqrt{3} \cdot m.$$

Hence, $\sin 60° = \sin POM = \frac{MP}{OP} = \frac{\sqrt{3} \cdot m}{2m} = \frac{\sqrt{3}}{2}$,

$$\cos 60° = \cos POM = \frac{OM}{OP} = \frac{m}{2m} = \frac{1}{2},$$

$$\tan 60° = \tan POM = \frac{MP}{OM} = \frac{\sqrt{3} \cdot m}{m} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

38. *To find the* **sine, cosine,** *and* **tangent** *of an angle of* **30°.**

With the same figure and construction as above we have the angle $OPM = 30°$, since it is half of OPQ, *i.e.* of 60°.

Hence, $\sin 30° = \sin OPM = \frac{MO}{PO} = \frac{m}{2m} = \frac{1}{2}$,

$$\cos 30° = \cos OPM = \frac{PM}{PO} = \frac{\sqrt{3} \cdot m}{2m} = \frac{\sqrt{3}}{2},$$

$$\tan 30° = \tan OPM = \frac{MO}{PM} = \frac{m}{\sqrt{3} \cdot m} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

39. *To find the* **sine, cosine,** *and* **tangent** *of an angle of* **0°.**

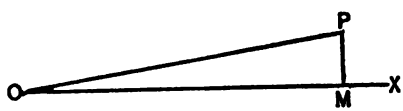


FIG. 18.

Let XOP be a small angle. Draw PM perpendicular to OX, and let OP be always of the same length, so that P lies on the circumference of a circle whose centre is O.

Then if the angle XOP be diminished, we can see that MP is diminished also, and that consequently $\frac{MP}{OP}$, which is $\sin XOP$, is diminished. And by diminishing the angle XOP sufficiently, *we can make MP as small as we please,* and therefore *we can make* $\sin XOP$ *smaller than any assignable number however small that number may be.*

This is what is meant when it is said that the value to which $\sin XOP$ approaches as the angle is diminished, is 0. This is expressed by saying,

$$\sin 0° = 0. \qquad \text{(i.)}$$

Again, as the angle XOP diminishes, OM approaches OP in length; and $\cos XOP$, which is $\frac{OM}{OP}$, approaches in value to $\frac{OP}{OP}$, *i.e.* to 1.

This is expressed by saying,

$$\cos 0^\circ = 1. \qquad \text{(ii.)}$$

Also, tan XOP is $\frac{MP}{OM}$; and we have seen that MP approaches 0, while OM does not; $\therefore$ tan XOP approaches 0.

This is expressed by saying,

$$\tan 0^\circ = 0. \qquad \text{(iii.)}$$

40. *To find the* **sine, cosine,** *and* **tangent** *of* **90°.**

Let XOY be a right angle $= 90^\circ$.

Draw XOP nearly a right angle; draw PM perpendicular to OX, and let OP be always of the same length, so that P lies on the circumference of a circle whose centre is O.

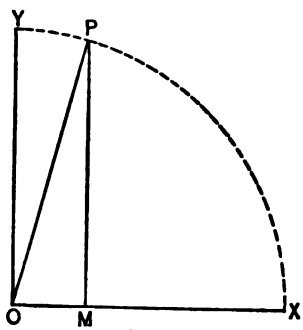


FIG. 19.

Then, as the angle XOP approaches to XOY, we can see that MP approaches OP, while OM continually diminishes.

Hence when XOP approaches 90°, sin XOP, which is $\frac{MP}{OP}$, approaches in value to $\frac{OP}{OP}$, that is to $\frac{1}{1}$, *i.e.* to 1.

Hence we say that $\sin 90^\circ = 1.$ (i.)

Again, when XOP approaches 90°,

cos XOP, which is $\frac{OM}{OP}$, approaches in value to $\frac{0}{OP}$, that is to 0.

Hence we say that $\cos 90^\circ = 0.$ (ii.)

Again, when XOP approaches 90°, tan XOP, which is $\frac{MP}{OM}$, approaches in value to $\frac{OP}{\text{a quantity which approaches } 0}$.

But in any fraction whose numerator does not diminish, the smaller the denominator, the greater the value of that fraction; and if the denominator continually diminishes, the value of the fraction continually increases.

Hence, *tan XOP can be made larger than any assigned number by making the angle XOP approach near enough to* 90°.

This is what we mean when we say that

$$\tan 90^\circ \text{ is } \textbf{infinity}, \text{ or, } \tan 90^\circ = \infty. \qquad \text{(iii.)}$$

41. The following table exhibits the above results:

angle	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
cosine	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞

The student may notice that the sine increases with the angle, while the cosine diminishes as the angle increases.

Also that the squares of the sines of 0°, 30°, 45°, 60°, and 90° are respectively 0, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, and $\frac{4}{4}$, and that the squares of the cosines of the same angles are $\frac{4}{4}$, $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$, and 0.

EXAMPLES. VI.

If $A = 90^\circ$, $B = 60^\circ$, $C = 30^\circ$, $D = 45^\circ$, prove the following:

1. $\cos^2 B - \sin^2 B = 1 - 2\sin^2 B$.
2. $\sin B \cdot \cos C + \sin C \cdot \cos B = \sin A$.
3. $2(\cos B \cdot \cos D + \sin B \cdot \sin D)^2 = 1 + \cos C$.
4. $2(\sin D \cdot \cos C - \sin C \cdot \cos D)^2 = 1 - \cos C$.
5. $\sin 30^\circ = .5$. 6. $\tan 60^\circ = 1.732\cdots$. 7. $\sin 45^\circ = .7071\cdots$.
8. $\sin 60^\circ = .8660\cdots$. 9. $\tan 30^\circ = .5773\cdots$.

CHAPTER VI

PRACTICAL APPLICATIONS

42. The actual measurement of the *line* joining two points which are any considerable distance apart, is a very tedious and difficult operation, especially when great accuracy is required; while the accurate measurement of an *angle* can, with proper instruments, be made with comparative ease and quickness.

43. A **Sextant** is an instrument for measuring the angle between the two lines drawn from the observer's eye to each of two distant objects respectively.

A **Theodolite** is an instrument for measuring angles in a horizontal plane; also for measuring "angles of elevation" and "angles of depression."

44. The angle made with the horizontal plane, by the line joining the observer's eye with a distant object, is called

(i.) its **angle of elevation,** when the object is *above* the observer;

(ii.) its **angle of depression,** when the object is *below* the observer.[1]

45. **Trigonometry** enables us, by measuring certain *angles,* to deduce other distances from one known distance, or, by the measurement of a *convenient* line, to deduce by the measurement of *angles* the lengths of lines whose actual measurement is difficult or impossible.

46. For this purpose we require the numerical values of the Trigonometrical Ratios of the angles observed. Accordingly, mathematical tables have been compiled, giving these ratios. These tables constitute a sort of numerical **Dictionary,** in which we can find the

[1] In measuring the angle of *depression* the telescope is turned from a horizontal position *downwards.* See Ex. VII. 3.

numerical values of the trigonometrical ratios of any required angle.

EXAMPLE 1. At a point 100 feet from the foot of a tower, the angle of elevation of the top of the tower is observed to be 60°. Find the *height* of the top of the tower above the point of observation.

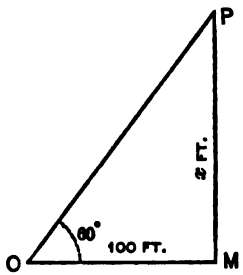


FIG. 20.

Let O be the point of observation; let P be the top of the tower; let a horizontal line through O meet the foot of the tower at the point M. Then $OM = 100$ feet, and the angle $MOP = 60°$. Let MP contain x feet.

Then $$\frac{MP}{OM} = \tan MOP = \tan 60° = \sqrt{3}.$$

$$\therefore \frac{x}{100} = \sqrt{3}.$$

$$\therefore x = 100 \,.\, \sqrt{3} = 100 \times 1.7320,$$
$$= 173.2.$$

Therefore the required height is 173.2.

EXAMPLE 2. At a point 100 yards from the foot of a building, I measure the angle of elevation of the top, and find that it is 23° 15′; what is the height of the building?

As in Example 1, let the height be x yards.

Then $\frac{x}{100} = \tan 23° 15'$.

From the table of tangents we find that $\tan 23° 15' = .4296339$.

Hence $x = 100 \times .4296339 = 42.96339$.

The height of the building = 43 yards, nearly. *Ans.*

EXAMPLE 3. A flagstaff, 25 feet high, stands on the top of a cliff; from a point on the seashore the angles of elevation of the highest and lowest points of the flagstaff are observed to be 47° 12′ and 45° 13′ respectively; find the height of the cliff.

Let O be the point of observation, PQ the flagstaff.

Let a horizontal line through O meet the vertical line PQ produced in M.

Then $QP = 25$ feet, $MOP = 47° 12'$, $MOQ = 45° 13'$.

Let $MQ = x$ feet; let $OM = y$ feet.

Then $\frac{MP}{OM} = \tan 47^\circ\ 12'$, $\therefore \frac{x+25}{y} = \tan 47^\circ\ 12'$,

and $\frac{MQ}{OM} = \tan 45^\circ\ 13'$, $\therefore \frac{x}{y} = \tan 45^\circ\ 13'$.

Hence, by division, $\therefore \frac{x+25}{x} = \frac{\tan 47^\circ\ 12'}{\tan 45^\circ\ 13'}$.

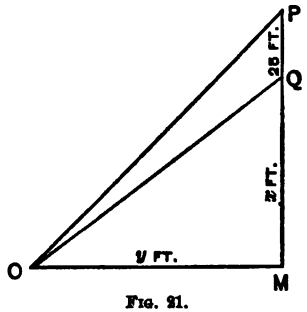


FIG. 21.

In the tables we find that

$$\tan 47^\circ\ 12' = 1.0799018, \text{ and } \tan 45^\circ\ 13' = 1.0075918.$$

$$\therefore 1 + \frac{25}{x} = \frac{1.0799018}{1.0075918} = 1 + \frac{.0723100}{1.0075918}.$$

$$\therefore \frac{x}{25} = \frac{1.0075918}{.0723100} = \frac{100759}{7231}.$$

$$\therefore x = \frac{2518975}{7231} = 348, \text{ nearly.}$$

Therefore the cliff is 348 feet high.

EXAMPLES. VII.

NOTE. The answers are given correct to three significant figures.

1. At a point 179 feet in a horizontal line from the foot of a column, the angle of elevation of the top of the column is observed to be 45°; what is the height of the column?

2. At a point 200 feet from, and on a level with the base of a tower, the angle of elevation of the top of the tower is observed to be 60°; what is the height of the tower?

3. From the top of a vertical cliff, the angle of depression of a point on the shore 150 feet from the base of the cliff, is observed to be 30°; find the height of the cliff.

4. From the top of a tower 117 feet high the angle of depression of the top of a house 37 feet high is observed to be 30°; how far is the top of the house from the tower?

5. A man 6 feet high stands at a distance of 4 ft. 9 in. from a lamp-post, and it is observed that his shadow is 19 feet long; find the height of the lamp.

6. The shadow of a tower in the sunlight is observed to be 100 feet long, and at the same time the shadow of a lamp-post 9 feet high is observed to be $3\sqrt{3}$ feet long. Find the angle of elevation of the sun, and the height of the tower.

7. From a point P on the bank of a river, just opposite a post Q on the other bank, a man walks at right angles to PQ to a point R so that PR is 100 yards; he then observes the angle PRQ to be $32^\circ\ 17'$; find the breadth of the river. ($\tan 32^\circ\ 17' = .6317667$.)

8. A fine wire 300 feet long is attached to the top of a spire and the inclination of the wire to the horizon when held tight is observed to be 40°; find the height of the spire. ($\sin 40^\circ = .6428$.)

9. A flagstaff 25 feet high stands on the top of a house; from a point on the plane on which the house stands the angles of elevation of the top and bottom of the flagstaff are observed to be 60° and 45° respectively; find the height of the house above the point of observation.

10. From the top of a cliff 100 feet high, the angles of depression of two ships at sea are observed to be 45° and 30° respectively; if the line joining the ships points directly to the foot of the cliff, find the distance between the ships.

11. A tower 100 feet high stands on the top of a cliff; from a point on the sand at the foot of the cliff the angles of elevation of the top and bottom of the tower are observed to be 75° and 60° respectively; find the height of the cliff. ($\tan 75^\circ = 2 + \sqrt{3}$).

12. A man walking along a straight road observes at one milestone a house in a direction making an angle 30° with the road, and that at the next milestone the angle is 60°; how far is the house from the road?

13. A man stands at a point A on the bank AB of a straight river and observes that the line joining A to a post C on the opposite bank makes with AB an angle of 30°. He then goes 400 yards along the bank to B and finds that BC makes with BA an angle of 60°; find the breadth of the river.

14. From the top of a hill the angles of depression of the top and bottom of a flagstaff 25 feet high at the foot of the hill are observed to be $45^\circ\ 13'$ and $47^\circ\ 12'$ respectively; find the height of the hill. ($\tan 45^\circ\ 13' = 1.0075918$. ($\tan 47^\circ\ 12' = 1.0799018$.)

15. An isosceles triangle of wood is placed on the ground in a vertical position facing the sun. If $2a$ be the base of the triangle, b its height, and 30° the altitude of the sun, find the tangent of half the angle at the apex of the shadow.

16. The length of the shadow of a vertical stick is to the length of the stick as $\sqrt{3} : 1$. If the stick be turned about its lower extremity in a vertical plane, so that the shadow is always in the same direction, find what will be the angle of its inclination to the horizon when the length of the shadow is the same as before.

17. What distance in space is travelled in an hour in consequence of the earth's rotation, by a person situated in latitude 60°? (Earth's radius = 4000 miles.)

CHAPTER VII

USE OF SIGNS + AND −. THE TRIGONOMETRIC RATIOS

47. Lines. The student has learned, in his study of algebra, that every quantitative symbol is affected by one of two signs, + or − ; that if b be such a symbol, then $+b$ and $-b$ are of the same absolute magnitude, but that they are *opposite* in *character; i.e. by whatever process* $+b$ *may have been generated,* $-b$ *has been generated by an exactly opposite process.* We express this symbolically thus:

$$a + b - b = a.$$

A line may be regarded as having been generated by the movement of a point. If, then, a line segment generated by the movement of a point in a given direction is represented by $+b$, where b is its measure, then a line segment generated by movement in the *opposite* direction is represented by $-b$, if b be *its* measure.

Thus line segments generated in opposite senses furnish an illustration of this algebraic convention.

48. We add line segments by placing the *beginning* of one segment at the *end* of another, so that all the extremities are collinear; *e.g.* suppose AB, CD, EF are line segments whose measures are a, b, c, respectively.

(1) Suppose AB, CD, EF are all generated in the same sense, then (1) Fig. 22 is the geometrical representation of their sum. The *measure* of their sum is $a + b + c$.

(2) Suppose AB and CD are generated in one sense and EF in the opposite sense, then (2) Fig. 22 is the geometrical representation of their sum. The measure of their sum is $a + b - c$.

(3) Suppose that AB and CD are generated in one sense, and that EF is generated in the opposite sense, and that the measure of $EF =$ the measure of CD; then (3) Fig. 22 is the geometrical rep-

resentation of their sum. The measure of their sum is $a + b - b$. In each case

$$AB + CD + EF = AF.$$

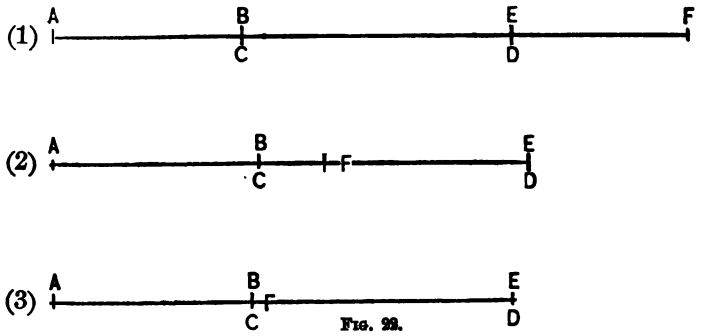


FIG. 22.

Any straight line, OX being given, then it follows that

Lines drawn parallel to OX in **one sense** may be assumed **positive**; that is, are *represented algebraically* by their *measures with the sign* + *before them;* then

Lines drawn parallel to OX in the **opposite sense** are **negative**; that is, are *represented algebraically* by their *measures with the sign* — *before them.*

49. In naming a line by the letters at its extremities, we can indicate by the **order of the letters** the direction in which the line is supposed to be drawn.

Hence in using the two letters at its extremities to represent a line, the student will find it advantageous always to pay careful attention to the **order of the letters.**

50. We should notice

(1) That it is immaterial which sense we *choose* as *positive.* But that, when once chosen, the *negative* sense is determined.

(2) That the line is the *result* of the movement, not the process.

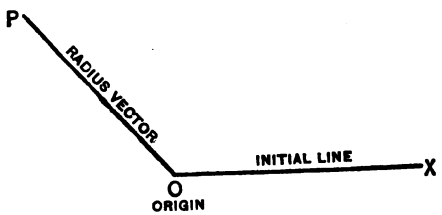


FIG. 23.

51. We have already said in Art. 5

(i.) that, when an angle XOP is described by OP turning about O in the direction *contrary to that of the hands of a watch*, the angle XOP is said to be **positive**; that is, *is represented algebraically* by its *measure with the sign* + *before it.*

(ii.) that, when an angle XOP is described by OP turning about O in the *same* direction as the hands of a watch, the angle is said to be **negative**; that is, *is represented algebraically* by its *measure with the sign* — *before it.*

52. Definitions. Quadrants.

Draw through any point O two lines OX, OY perpendicular to each other.

(OX is drawn usually horizontally.)

The plane is divided into four parts called **quadrants.**

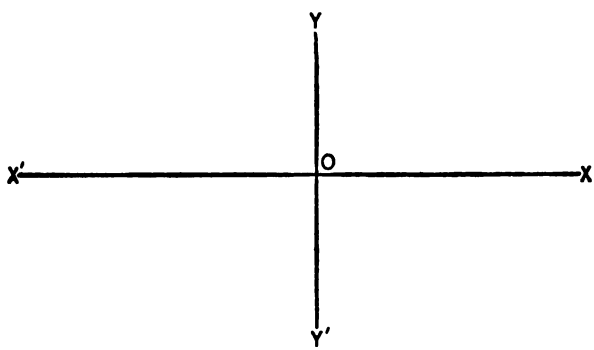


Fig. 24.

The part of the plane lying between

(i.) OX and OY constitutes the *first* **quadrant.**

(ii.) OY and OX' constitutes the *second* **quadrant.**

(iii.) OX' and OY' constitutes the *third* **quadrant.**

(iv.) OY' and OX constitutes the *fourth* **quadrant.**

An **angle** is said to be of the **first quadrant** if the radius vector OP is in the first quadrant; of the second, third, or fourth quadrant according as it is in the second, third, or fourth quadrant.

53. Let A be an angle between 0° and 90°, and let n be any *whole* number, positive, negative, or zero. Then

(i.) $2n \times 180° + A$ represents algebraically an angle of the *first* quadrant.

(ii.) $2n \times 180° - A$ represents algebraically an angle of the *fourth* quadrant.

[For $2n \times 180°$ represents some number n of *complete* revolutions of OP; so that after describing $n \times 360°$, OP is again in the position OX.]

(iii.) $(2n+1) \times 180° - A$ represents algebraically an angle of the *second* quadrant.

(iv.) $(2n+1) \times 180° + A$ represents algebraically an angle of the *third* quadrant.

[For after describing $(2n+1) \times 180°$, OP is in the position OX.]

The corresponding expressions in circular measure are

(i.) $2n\pi+\theta$; (ii.) $2n\pi-\theta$; (iii.) $(2n+1)\pi-\theta$; (iv.) $(2n+1)\pi+\theta$.

EXAMPLES. VIII.

Draw a figure giving the position of the radius vector after it has turned through each of the following angles, and state to what quadrant each angle belongs.

1. $270°$.
2. $370°$.
3. $425°$.
4. $590°$.
5. $-30°$.
6. $-330°$.
7. $-480°$.
8. $-750°$.
9. $\frac{27\pi}{4}$.
10. $2n\pi+\frac{\pi}{6}$.
11. $(2n+1)\pi+\frac{\pi}{3}$.
12. $(2n+1)\pi-\frac{\pi}{4}$.
13. $2n\pi-\frac{\pi}{2}$.
14. $(2n+1)\pi-\frac{\pi}{2}$.

NOTE. $n\pi$ always stands for a whole number of two right angles.

54. For the remainder of our discussion, we shall make the following agreements:

Choose O as origin; choose OX, drawn horizontally, as initial line.

I. For lines parallel to the initial line OX, the positive direction is from O to X.

It is convenient, when considering one angle only, so to arrange the figure that from O to X is from left to right.

We next *choose* the positive direction of revolution about O for the radius vector.

II. For lines perpendicular to OX, the positive direction is from O to Y, where XOY is a *positive right angle.*

When from O to X is from left to right, and the positive direction of revolution about O is contrary to that of the hands of a watch with its face

upwards (counter-clockwise), then from O to Y is in the direction from the bottom of the page to the top.

III. For lines parallel to OP, the positive direction is from O to P.

The radius vector carries its positive direction around with it; hence OP is *always positive*. The direction from O to P, as OP revolves, nowhere undergoes a sudden reversal of direction such as is indicated by a change of sign.

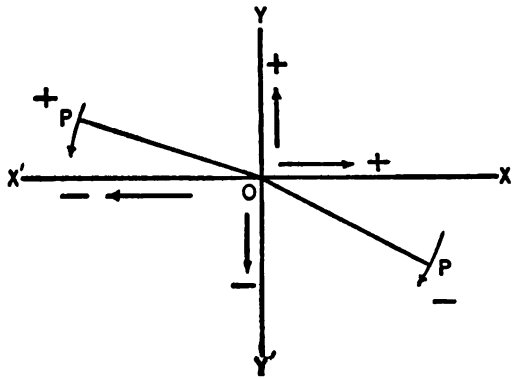


FIG. 25.

As already explained, the *negative* direction is in all cases exactly the reverse of the positive direction.

We have said that the definitions of the Trigonometrical Ratios (Art. 26) apply to angles of any magnitude. We may say also that when the angle exceeds a right angle, a line on which the point P is taken must be considered the radius vector; and that the order of the letters in MP, OM, OP give the directions of the lines, and therefore the *signs* of their measures.

55. We will now show that the trigonometrical ratios of an angle vary in Sign according to the Quadrant in which the radius vector of the angle happens to be.

From the definition we have, with the usual letters,

$$\sin XOP = \frac{MP}{OP}, \quad \cos XOP = \frac{OM}{OP}, \quad \tan XOP = \frac{MP}{OM}.$$

I. When OP is in the first quadrant (Fig. 26),

MP is *positive* because from M to P is *upwards* (Art. 54, II.);

OM is *positive* because from O to M is *towards the right* (Art. 54, I.);

OP is *positive* (Art. 54, III.).

Hence, if A be any angle of the *first* quadrant,

$$\sin A, \text{ which is } \frac{MP}{OP}, \text{ is } \textit{positive};$$

$$\cos A, \text{ which is } \frac{OM}{OP}, \text{ is } \textit{positive};$$

$$\tan A, \text{ which is } \frac{MP}{OM}, \text{ is } \textit{positive}.$$

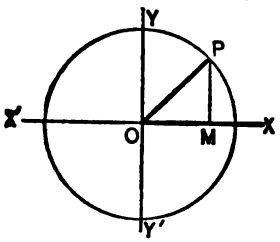


FIG. 26.

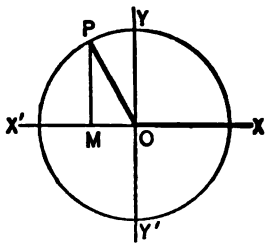


FIG. 27.

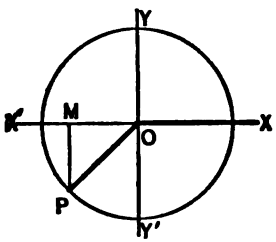


FIG. 28.

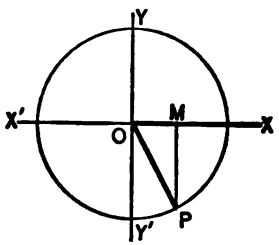


FIG. 29.

II. When OP is in the **second** quadrant (Fig. 27),
MP is *positive*, because from M to P is *upwards*.
OM is *negative*, because from O to M is *towards the left*.
OP is *positive*.
Hence, if A be any angle of the *second* quadrant,

$$\sin A, \text{ which is } \frac{MP}{OP}, \text{ is } \textit{positive};$$

$$\cos A, \text{ which is } \frac{OM}{OP}, \text{ is } \textit{negative};$$

$$\tan A, \text{ which is } \frac{MP}{OM}, \text{ is } \textit{negative}.$$

III. When OP is in the **third** quadrant (Fig. 28), MP is *negative*, OM is *negative*, OP is *positive*.

So that, if A be any angle of the *third* quadrant,

$\sin A$ is *negative*, $\cos A$ is *negative*, $\tan A$ is *positive*.

IV. When OP lies in the **fourth** quadrant (Fig. 29), MP is *negative*, OM is *positive*, OP is *positive*.

So that, if A be any angle of the *fourth* quadrant,

$\sin A$ is *negative*, $\cos A$ is *positive*, $\tan A$ is *negative*.

56. The table given below exhibits the results of the last article:

Quadrant . . .	I.	II.	III.	IV.
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent . . .	+	−	+	−

The student should notice that for any particular quadrant the three *signs* of sine, cosine, and tangent are unlike their signs for any other quadrant.

57. The cosecant, secant, and cotangent of an angle A have the same *sign* as the sine, cosine, and tangent of A respectively.

EXAMPLES. IX.

State the *sign* of the sine, cosine, and tangent of each of the following angles:

1. $60°$.	2. $135°$.	3. $265°$.
4. $275°$.	5. $-10°$.	6. $-91°$.
7. $-193°$.	8. $-350°$.	9. $-1000°$.
10. $2n\pi + \frac{1}{4}\pi$.	11. $2n\pi + \frac{3}{4}\pi$.	12. $2n\pi - \frac{1}{6}\pi$.

58. *The ratios of an angle of any quadrant may be expressed in terms of the ratios of a positive angle between* $0°$ *and* $90°$.

The algebraic values of the ratios depend only on the position of the radius vector.

I. Hence the proposition is evident for angles of the first quadrant.

II. Consider an angle of the second quadrant.

Let XOP_2 be an angle of the second quadrant. With OP_2 as radius describe a circle about O as centre. Construct $\angle XOP_1$, P_1 being on the circumference of the circle [see I., Art. 32], so that

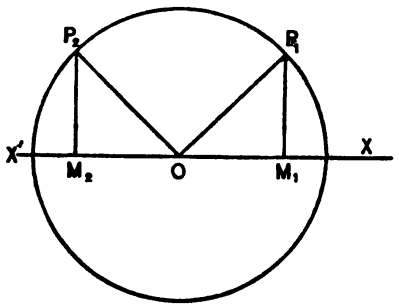


FIG. 30.

the measure of $\angle XOP_1 =$ measure of $\angle X'OP_2$. From P_1 and P_2 let perpendiculars fall on XX', cutting it in M_1 and M_2 respectively.

Then $$\triangle M_1OP_1 = \triangle M_2OP_2;$$

whence $$M_2P_2 = M_1P_1,$$

and $$OM_2 = -\,OM_1 \text{ (Art. 47).}$$

$$\sin XOP_2 = \frac{M_2P_2}{OP_2};\ \sin XOP_1 = \frac{M_1P_1}{OP_1}.$$

But $$\frac{M_2P_2}{OP_2} = \frac{M_1P_1}{OP_1};$$

$$\therefore\ \sin XOP_2 = \sin XOP_1;$$

$$\cos XOP_2 = \frac{OM_2}{OP_2} = -\frac{OM_1}{OP_1} = -\cos XOP_1;$$

$$\tan XOP_2 = \frac{M_2P_2}{OM_2} = -\frac{M_1P_1}{OM_1} = -\tan XOP_1.$$

III. Angles of third quadrant.

Let OXP_3 be an angle of the third quadrant (Fig. 31).

By reasoning similar to that in II.,

$$M_3P_3 = -\,M_1P_1;\ OM_3 = -\,OM_1 \text{ (Art. 47).}$$

Hence
$$\sin XOP_3 = \frac{M_3P_3}{OP_3} = -\frac{M_1P_1}{OP_1} = -\sin XOP_1;$$
$$\cos XOP_3 = \frac{OM_3}{OP_3} = -\frac{OM_1}{OP_1} = -\cos XOP_1;$$
$$\tan XOP_3 = -\frac{M_3P_3}{OM_3} = \frac{M_1P_1}{OM_1} = \tan XOP_1.$$

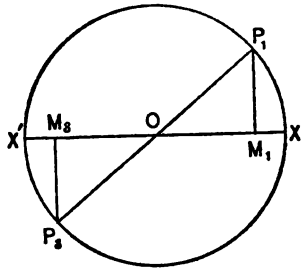


FIG. 31.

IV. We leave as an exercise for the student to prove: If XOP_4 is an angle of the fourth quadrant,
$$\sin XOP_4 = -\sin XOP_1;$$
$$\cos XOP_4 = \cos XOP_1;$$
$$\tan XOP_4 = -\tan XOP_1.$$

We have now proved our theorem.

COROLLARY I. The results of the preceding article may be formulated as follows (see Art. 53):

$$\left.\begin{aligned} \sin(2n\,180^\circ \pm A) &= \pm \sin A; \\ \cos(2n\,180^\circ \pm A) &= \cos A; \\ \tan(2n\,180^\circ \pm A) &= \pm \tan A. \end{aligned}\right\} \text{I.}$$

$$\left.\begin{aligned} \sin[(2n+1)\,180^\circ \pm A] &= \mp \sin A; \\ \cos[(2n+1)\,180^\circ \pm A] &= -\cos A; \\ \tan[(2n+1)\,180^\circ \pm A] &= \pm \tan A. \end{aligned}\right\} \text{II.}$$

where A is an angle between 0° and 90°, and n is any integer positive, or negative, or zero, and either the *upper* sign or the *lower* sign is read on both sides of the equations.

COROLLARY II. Making $n = 1$ in each of the equations I., we obtain the very important formulæ:

$$\sin(360° \pm A) = \pm \sin A;$$
$$\cos(360° \pm A) = \cos A;$$
$$\tan(360° \pm A) = \pm \tan A.$$

Making $n = 0$ in II.,

$$\sin(180° \pm A) = \mp \sin A;$$
$$\cos(180° \pm A) = -\cos A;$$
$$\tan(180° \pm A) = \pm \tan A.$$

Making $n = 0$ in I., and reading the lower sign, we obtain:

$$\sin(-A) = -\sin A;$$
$$\cos(-A) = \cos A;$$
$$\tan(-A) = -\tan A.$$

59. *If A be any angle,*

$$\sin(180° - A) = \sin A;$$
$$\cos(180° - A) = -\cos A;$$
$$\tan(180° - A) = -\tan A.$$

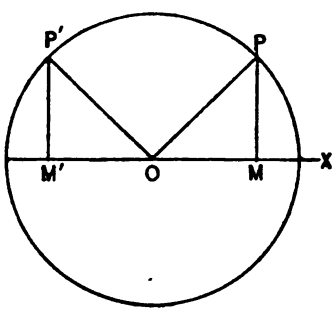


FIG. 32.

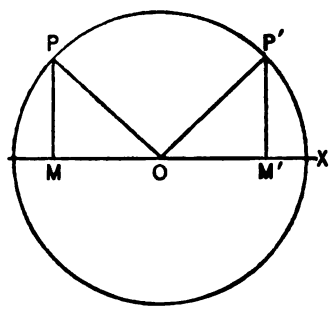


FIG. 33.

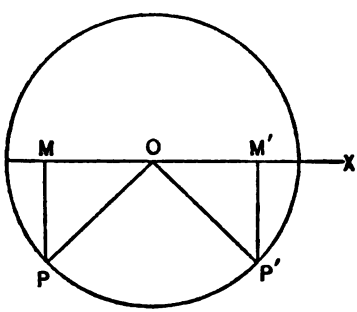


FIG. 34.

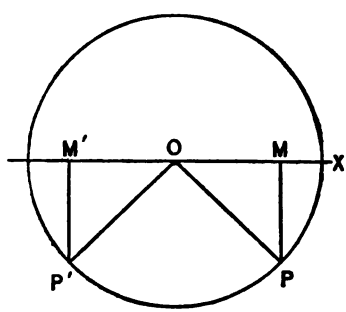


FIG. 35.

Let XOP be any angle, A; see Figs. 32, 33, 34, 35.

Let XOP' be $180° - A$.

It is easily proved that

if A is an angle of 1st quadrant, $180° - A$ is an angle of 2d quadrant;

if A is an angle of 2d quadrant, $180° - A$ is an angle of 1st quadrant;

if A is an angle of 3d quadrant, $180° - A$ is an angle of 4th quadrant;

if A is an angle of 4th quadrant, $180° - A$ is an angle of 3d quadrant;

and as in Art. 58, $$P'M' = PM;$$

$$OM' = -OM.$$

In each case

$$\sin(180° - A) = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin A;$$

$$\cos(180° - A) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A;$$

$$\tan(180° - A) = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan A.$$ Q.E.D.

60. Definition. One angle is said to be the **supplement** of another when their sum is two right angles. The results of Art. 58 may now be stated in words:

The *sine* of the *supplement* of an *angle* = the *sine* of the *angle*.

The *cosine* of the *supplement* of an *angle* = the *negative cosine* of the *angle*.

The *tangent* of the *supplement* of an *angle* = the *negative tangent* of the *angle*.

EXAMPLES. X.

Prove, drawing a separate figure in each case, that

1. $\sin 60° = \sin 120°$.
2. $\sin 340° = \sin(-160°)$.
3. $\sin(-40°) = \sin 220°$.
4. $\cos 320° = -\cos(-140°)$.
5. $\cos(-380°) = -\cos 560°$.
6. $\cos 195° = -\cos(-15°)$.

If A, B, C be the angles of a triangle, prove

7. $\sin A = \sin(B + C)$.
8. $\sin C = \sin(A + B)$.
9. $\cos B = -\cos(A + C)$.
10. $\cos A = -\cos(C + B)$.

Prove by means of a figure that

11. $\sin(-A) = -\sin A$.
12. $\cos(-A) = \cos A$.

DEFINITION. One angle is said to be the **complement** of another if their **sum** = a **right angle.**

EXAMPLE. The complement of 190° is − 100°.

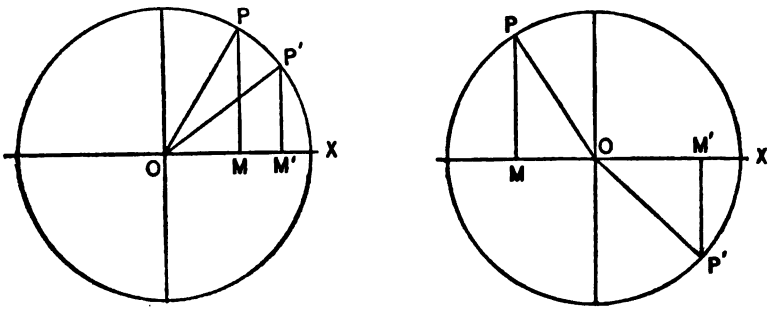


FIG. 36.

61. *To prove if A is an angle, that*

$$\sin(90° - A) = \cos A;$$
$$\cos(90° - A) = \sin A;$$
$$\tan(90° - A) = \cot A.$$

We shall prove the theorem in case A is an angle of the first or of the second quadrant. The student should prove it for the remaining cases.

If A is an angle of the 1st quadrant, $90° - A$ is an angle of the 1st quadrant.

If A is an angle of the 2d quadrant, $90° - A$ is an angle of the 4th quadrant.

Let $XOP = A$, and $XOP' = 90° - A$.

Then, triangles MPO and $M'OP'$ are similar (Fig. 36).

$$\frac{M'P'}{OP'} = \frac{OM}{OP} \text{ (i.)}, \quad \text{and } \frac{OM'}{OP'} = \frac{MP}{OP} \text{ (ii.)}, \quad \text{and } \frac{M'P'}{OM'} = \frac{OM}{MP} \text{ (iii.)};$$

or

$$\sin(90° - A) = \cos A;$$
$$\cos(90° - A) = \sin A;$$
$$\tan(90° - A) = \cot A.$$

EXAMPLES. XI.

Find the complements of

1. 30°.
2. 190°.
3. 90°.
4. 350°.
5. − 25°.
6. − 320°.
7. $\frac{1}{4}\pi$.
8. $-\frac{1}{6}\pi$.

Prove by drawing a figure in each case:

9. $\sin 70° = \cos 20°$.
10. $\cos 47° 16' = \sin 42° 44'$.
11. $\tan 79° = \cot 11°$.
12. $\sec 36° = \operatorname{cosec} 54°$.

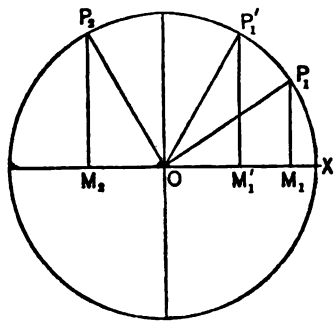
13. $\cos 105° = \sin - 15°$.
14. $\tan 135° = \cot - 45°$.
15. $\sec A = \operatorname{cosec}(90° - A)$.
16. $\cot A = \tan(90° - A)$.

If A, B, C be the angles of a triangle, so that $A + B + C = 180°$, prove

17 $\cos \frac{1}{2} A = \sin \frac{1}{2}(B + C)$.
18. $\cos \frac{1}{2} B = \sin \frac{1}{2}(A + C)$.
19. $\sin \frac{1}{2} C = \cos \frac{1}{2}(A + B)$.
20. $\sin \frac{1}{2} A = \cos \frac{1}{2}(B + C)$.

62. *To find the ratios of* $90° + A$.

We shall consider only the case when A is between 0° and 90°.



Let $XOP_1 = A$, and $XOP_2 = 90° + A$.

Construct $XOP'_1 = 90° - A$.

Then the measure of $XOP_2 =$ measure of $180° - XOP'_1$.

$$\therefore \sin XOP_2 = \sin XOP'_1 \text{ (Art. 57)}.$$

But $$\sin XOP'_1 = \cos XOP_1 \text{ (Art. 60)};$$

i.e. $$\sin(90° + A) = \cos A, \text{ and similarly}$$

$$\cos(90° + A) = -\sin A,$$

$$\tan(90° + A) = -\cot A.$$

The proof for the remaining cases offers little difficulty.

EXAMPLES. XII.

Find the algebraical value of the sine, cosine, and tangent of the following angles:

1. 150°.
2. 135°.
3. − 240°.
4. 330°.
5. − 45°.
6. − 300°.
7. 225°.
8. − 135°.
9. 390°.
10. 750°.
11. − 840°.
12. 1020°.
13. $2n\pi + \frac{\pi}{4}$.
14. $(2n + 1)\pi - \frac{\pi}{3}$.
15. $(2n - 1)\pi + \frac{\pi}{6}$.
16. $\sin A = \frac{1}{2}$.
17. $\sin A = \frac{1}{\sqrt{2}}$.
18. $\sin A = \frac{\sqrt{3}}{2}$.
19. $\sin A = -\frac{1}{2}$.

Find four angles between zero and + 8 right angles which satisfy the equations:

20. $\sin A = \sin 20°$ **21.** $\sin\theta = -\frac{1}{\sqrt{2}}$. **22.** $\sin\theta = -\sin\frac{\pi}{7}$.

23. Prove that 30°, 150°, − 330°, 390°, − 210° have the same sine.

24. Show that each of the following angles has the same cosine:

$$-120°,\ 240°,\ 480°,\ -480°.$$

25. The angles 60° and − 120° have one of the Trigonometrical Ratios the same for both; which of the ratios is it?

26. Can the following angles have any one of their Trigonometrical Ratios the same for all? − 23°, 157°, and − 157°.

63. *The ratios of the angles*

$$2n\pi,\ 2n\pi + \frac{\pi}{2},\ (2n+1)\pi,\ (2n+1)\pi + \frac{\pi}{2}.$$

The reasoning by which we obtained the ratios of 0° and 90° applies with equal force for $2n\pi$ and $2n\pi + \frac{\pi}{2}$ respectively, whence we obtain,

$$\sin 2n\pi = 0, \quad \cos 2n\pi = 1, \quad \tan 2n\pi = 0; \qquad \text{(i.)}$$

$$\sin\left(2n\pi + \frac{\pi}{2}\right) = 1, \quad \cos\left(2n\pi + \frac{\pi}{2}\right) = 0, \quad \tan\left(2n\pi + \frac{\pi}{2}\right) = \infty. \qquad \text{(ii.)}$$

We will now obtain the ratios of $(2n+1)\pi$.

We have shown that

$$\sin XOP_2 = \sin XOP_1; \qquad \text{(Fig. 30, Art. 58.)}$$
$$\cos XOP_2 = -\cos XOP_1;$$
$$\tan XOP_2 = -\tan XOP_1,$$

however little the difference between 180° and XOP_2 may be. If this difference be very small, then XOP_1 differs from 0° by a very small quantity. Hence

$$\sin XOP_1 = 0, \quad \cos XOP_1 = 1, \quad \tan XOP_1 = 0;$$

whence $\sin XOP_2 = 0, \quad \cos XOP_2 = -1, \quad \tan XOP_2 = 0;$

or $\sin(2n+1)\pi = 0, \ \cos(2n+1)\pi = -1, \ \tan(2n+1)\pi = 0.$ (iii.)

We leave as an exercise to prove that

$$\left.\begin{aligned} &\sin\left[(2n+1)\pi + \frac{\pi}{2}\right] = -1, \quad \cos\left[(2n+1)\pi + \frac{\pi}{2}\right] = 0, \\ &\tan\left[(2n+1)\pi + \frac{\pi}{2}\right] = \infty.^* \end{aligned}\right\} \qquad \text{(iv.)}$$

* See remark under corollary, next page.

COROLLARY. Making (1) $n = 0$ in each of the preceding formulæ, and (2) $n = 1$ in I., we obtain:

Angle	0°	90°	180°	270°	360°
Sine	0	1	0	−1	0
Cosine	1	0	−1	0	1
Tangent . . .	0	∞	0	∞	0

The sign has been omitted from ∞, since all that we know is, when the angle is a little less than 90° or 270°, or a little greater than 90° or 270°, that the tangent is very large *numerically;* that in the first case, viz. the angle is a little less than either 90° or 270°, the tangent is positive; and in the second case, negative.

64. By Art. 32, if A be an angle of the first quadrant,

(1) A small change in A will produce a change in the values of its ratios.

(2) So long as the angle A is unchanged, its ratios are unchanged.

(3) To each value of A there is a definite value for each of its ratios. By Art. 58 the ratios of an angle of *any* quadrant may be expressed in terms of the ratios of an angle of the first quadrant. Hence the theorems of Art. 32 are true for any angle; or, more briefly, *the ratios of an angle are unique.*

DEFINITION. When two quantities are so related that a change in the value of one of them produces a corresponding change in the value of the other, the second is said to be a function of the first.

It appears from this article that the angle and its ratios are so related. For this reason the trigonometrical ratios are generally called trigonometric functions or circular functions.

65. To trace the changes in the magnitude and sign of sin A as A increases from 0° to 360°.

Construct circles with OP as radius. Let PM be perpendicular to OX. Let $A = XOP$.

Then $$\sin A = \frac{MP}{OP}.$$

When the angle A is 0°, MP is zero, and when A is 90°, MP is equal to OP; and as A continuously increases from 0° to 90°, MP

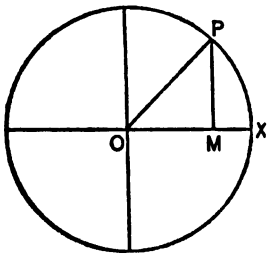


Fig. 37.

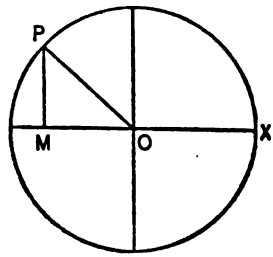


Fig. 38.

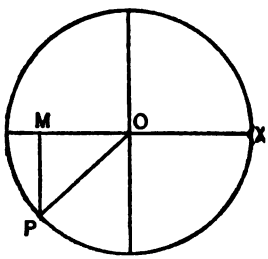


Fig. 39.

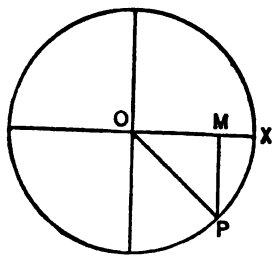


Fig. 40.

increases continuously from zero to OP; also OP is always equal to OX.

Therefore, when $A = 0°$, the fraction $\frac{MP}{OP}$ is equal to $\frac{0}{OP}$, that is 0; when $A = 90°$, the fraction $\frac{MP}{OP}$ is equal to $\frac{OP}{OP}$, that is 1; and as A continuously increases from 0° to 90°, the *numerator* of the fraction $\frac{MP}{OP}$ *continuously increases* from zero to OP, while the denominator is unchanged, and therefore the *fraction* $\frac{MP}{OP}$, which is sin A, *increases continuously* from 0 to 1, and is positive.

As A increases from 0° to 90°, MP increases from zero to OP, and is positive.

Therefore sin A increases from 0 to 1, and is positive.

As A increases from 90° to 180°, MP decreases from OP to zero, and is positive.

Therefore sin A decreases from 1 to 0, and is positive.

As A increases from 180° to 270°, MP increases numerically from 0 to OP, and is negative; hence sin A increases numerically from 0 to 1, and is negative.

As A increases from 270° to 360°, MP decreases from OP to zero, and is negative.

Therefore sin A decreases numerically from 1 to 0, and is negative.

COROLLARY. Therefore we may conclude that **sin A** is **never greater than 1**; that **cos A** is **never greater than 1**. That the *numerical* value of sec A or of cosec A is never less than 1. For

$$\sec A = \frac{1}{\cos A}, \quad \operatorname{cosec} A = \frac{1}{\sin A}. \qquad \text{(Def.)}$$

EXAMPLES. XIII.

Trace the changes in sign and magnitude as A increases from 0° to 360° of

1. $\cos A$.
2. $\tan A$.
3. $\cot A$.
4. $\sec A$.
5. $\operatorname{cosec} A$.
6. $1 - \sin A$.
7. $\sin^2 A$.
8. $\sin A \cdot \cos A$.

CHAPTER VIII

ON THE RELATIONS BETWEEN THE TRIGONOMETRIC RATIOS

66.

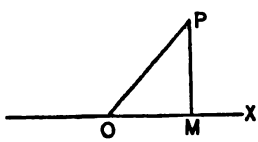


FIG. 41.

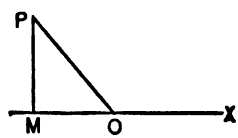


FIG. 42.

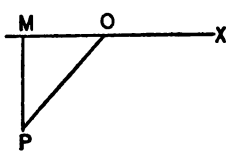


FIG. 43.

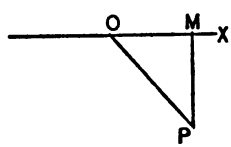


FIG. 44.

Let $XOP = A$, any angle. (The figures are those of Art. 25.)
The following relations are evident from the definitions:

$$\operatorname{cosec} A = \frac{1}{\sin A}; \tag{1}$$

$$\sec A = \frac{1}{\cos A}; \tag{2}$$

$$\cot A = \frac{1}{\tan A}. \tag{3}$$

To prove $\tan A = \dfrac{\sin A}{\cos A}$.

$$\sin A = \frac{MP}{OP}, \quad \cos A = \frac{OM}{OP}.$$

$$\therefore \frac{\sin A}{\cos A} = \frac{\dfrac{MP}{OP}}{\dfrac{OM}{OP}} = \frac{MP}{OM} = \tan A. \tag{4}$$ Q.E.D.

Similarly $\dfrac{\cos A}{\sin A} = \cot A.$ (5)

67. To prove $\sin^2 A + \cos^2 A = 1$.

$$\sin^2 A = \frac{MP^2}{OP^2}, \quad \cos^2 A = \frac{OM^2}{OP^2}.$$

Now $\sin^2 A + \cos^2 A = \dfrac{MP^2}{OP^2} + \dfrac{OM^2}{OP^2} = \dfrac{MP^2 + OM^2}{OP^2} = \dfrac{OP^2}{OP^2} = 1,$

since $\qquad MP^2 + OM^2 = OP^2. \qquad$ (Pythagorean Prop.)

Similarly we may prove that

$$1 + \tan^2 A = \sec^2 A,$$

and that

$$1 + \cot^2 A = \operatorname{cosec}^2 A.$$

68. The following is a **List of Formulæ** with which the student must make himself familiar:

$$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sec A = \frac{1}{\cos A},$$

$$\cot A = \frac{1}{\tan A}, \quad \tan A = \frac{\sin A}{\cos A}, \quad \cot A = \frac{\cos A}{\sin A},$$

$$\sin^2 A + \cos^2 A = 1,$$

$$\tan^2 A + 1 = \sec^2 A,$$

$$\cot^2 A + 1 = \operatorname{cosec}^2 A.$$

69. By means of these formulæ we are able to transform a given trigonometrical expression into a great variety of equivalent expressions.

EXAMPLE. Prove that $\tan A + \cot A = \sec A \cdot \operatorname{cosec} A$.

Since $\quad \tan A = \dfrac{\sin A}{\cos A}, \quad \cot A = \dfrac{\cos A}{\sin A}, \quad \sec A = \dfrac{1}{\cos A}, \quad \operatorname{cosec} A = \dfrac{1}{\sin A},$

we have

$$\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\cos A \cdot \sin A} = \frac{1}{\cos A \cdot \sin A} = \sec A \cdot \operatorname{cosec} A. \qquad \text{[Art. 67.]}$$

70. It is sometimes convenient to write a given expression in terms of the *sine* only, or in terms of the *cosine* only.

EXAMPLE I. Prove that $\sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta = 1 - \cos^4 \theta$.

By Art. 67, we have $\quad \sin^2 \theta = 1 - \cos^2 \theta,$

hence

$$\begin{aligned} \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta &= (1 - \cos^2 \theta)^2 + 2(1 - \cos^2 \theta) \times \cos^2 \theta \\ &= (1 - 2\cos^2 \theta + \cos^4 \theta) + (2\cos^2 \theta - 2\cos^4 \theta) \\ &= 1 - \cos^4 \theta. \end{aligned}$$

Example II. Express $\sin^4\theta + \cos^4\theta$ in terms of $\cos\theta$.

$$\begin{aligned}\sin^4\theta + \cos^4\theta &= (1 - \cos^2\theta)^2 + \cos^4\theta \\ &= (1 - 2\cos^2\theta + \cos^4\theta) + \cos^4\theta \\ &= 1 - 2\cos^2\theta + 2\cos^4\theta.\end{aligned}$$

Note. $(1 - \cos\theta)$ is called the **versed sine of θ**, and is written versin θ.

EXAMPLES. XIV.

Prove the following statements:

1. $\cos A \cdot \tan A = \sin A$.
2. $\cot A \cdot \tan A = 1$.
3. $\cos A = \sin A \cdot \cot A$.
4. $\sec A \cdot \cot A = \operatorname{cosec} A$.
5. $\operatorname{cosec} A \cdot \tan A = \sec A$.
6. $(\tan A + \cot A)\sin A \cdot \cos A = 1$.
7. $(\tan A - \cot A)\sin A \cdot \cos A = \sin^2 A - \cos^2 A$.
8. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.
9. $(\sin A + \cos A)^2 = 1 + 2\sin A \cdot \cos A$.
10. $(\sin A - \cos A)^2 = 1 - 2\sin A \cdot \cos A$.
11. $\cos^4 B - \sin^4 B = 2\cos^2 B - 1$.
12. $(\sin^2 B + \cos^2 B)^2 = 1$.
13. $(\sin^2 B - \cos^2 B)^2 = 1 - 4\cos^2 B + 4\cos^4 B$.
14. $1 - \tan^4 B = 2\sec^2 B - \sec^4 B$.
15. $(\sec B - \tan B)(\sec B + \tan B) = 1$.
16. $(\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta) = 1$.
17. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$.
18. $\cos^3\theta - \sin^3\theta = (\cos\theta - \sin\theta)(1 + \sin\theta\cos\theta)$.
19. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta \cdot \cos^2\theta$.
20. $(\sin^6\theta - \cos^6\theta) = (2\sin^2\theta - 1)(1 - \sin^2\theta + \sin^4\theta)$.
21. $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \cdot \tan B$.
22. $\dfrac{\cot\alpha + \tan\beta}{\tan\alpha + \cot\beta} = \cot\alpha \cdot \tan\beta$.
23. $\dfrac{1 - \sin A}{1 + \sin A} = (\sec A - \tan A)^2$.
24. $\dfrac{1 + \cos A}{1 - \cos A} = (\operatorname{cosec} A + \cot A)^2$.
25. $2\,\text{versin}\,\theta - \text{versin}^2\theta = \sin^2\theta$.
26. $\text{versin}\,\theta\,(1 + \cos\theta) = \sin^2\theta$.

Express in terms of (i.) $\cos\theta$, (ii.) of $\sin\theta$,

27. $\cos^4\theta - \sin^4\theta$.
28. $(\sin^2\theta - \cos^2\theta)^2$.
29. $1 - \tan^4\theta$.
30. $\sin^6\theta + \cos^6\theta$.
31. $\tan^2\theta + \cot^2\theta$.
32. $1 + \cot^4\theta$.
33. $1 + \cot^2\theta - \operatorname{cosec}^2\theta$.
34. $2\tan^4\theta - 4\sin^2\theta$.
35. Show $\cos(90° - \alpha)\sin(180° - \alpha) - \sin(90° - \alpha)\cos(180° - \alpha) = 1$.
36. $\tan(180° + 45°)\tan 45° + 1 = \sec 245°$.
37. $\cot 120° \cdot \cot(-60°) + 1 = \dfrac{1}{\sin^2 60°} = \dfrac{1}{\cos^2 30°}$.
38. $\cos^2[(2n + 1)\pi + A] = \dfrac{\cos^2 A + \sin^2(1800 - A)}{\sec^2 A}$.

By the aid of the formulæ of Art. 68 we may express the ratios of any angle in terms of any *one* of its ratios, *e.g.*

To express the other trigonometric ratios of θ in terms of its tangent.

$$\cot\theta = \frac{1}{\tan\theta}.$$

$$\sec\theta = \pm\sqrt{1+\tan^2\theta}.$$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\pm\sqrt{1+\tan^2\theta}}.$$

$$\sin\theta = \tan\theta\ \cos\theta = \frac{\tan\theta}{\pm\sqrt{\tan^2\theta+1}}.$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\pm\sqrt{1+\tan^2\theta}}{\tan\theta}.$$

71. If θ be less than 90°, the following method can be employed with advantage:

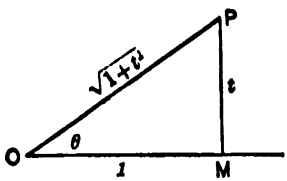


FIG. 45.

Let MOP be a right triangle; Fig. 45,

Let $MP = t$, $OM = 1$, $\therefore OP = \sqrt{1+t^2}$.

$$\tan\theta = \frac{t}{1} = t,\ \sin\theta = \frac{t}{\sqrt{1+t^2}},$$

$\cos\theta = \dfrac{1}{\sqrt{1+t^2}}$, and so on. With suitable modifications we may employ this method whatever be the magnitude of θ.

Hence if we are given $\tan\theta$ we can calculate the value of the remaining ratios by either of the above methods.

E.g. suppose $\tan\theta = \sqrt{3}$, then

$$\sin\theta = \frac{\sqrt{3}}{\sqrt{1+3}} = \frac{\sqrt{3}}{2}, \text{ and so on.}$$

72. The expressions obtained for $\sin\theta$, $\cos\theta$, $\sec\theta$, and $\csc\theta$, in Art. 70, are affected with a double sign. Hence these ratios are not *uniquely* determined. If, however, we *know tan θ* and in addition to *what quadrant θ belongs*, we can determine the respective signs of the ratios named.

E.g. if $\tan\theta = \sqrt{3}$ and we know that θ is of the third quadrant, then $\sin\theta = -\dfrac{\sqrt{3}}{2}$.

EXAMPLES. XV. (*a*).

1. Express all the other ratios of A in terms of $\cos A$.

2. Express all the ratios of $90° + A$ in terms of $\cot A$; $A < 90°$.

3. Express all the ratios of $180° - A$ in terms of $\sec A$.

4. Express all the other ratios of A in terms of $\csc A$.

5. Use formulæ of Art. 67 to express all the other ratios of A in terms of $\sin A$.

6. Use the method of Art. 71 to express all the ratios of A in terms of cosine A; $A < 90°$.

EXAMPLES. XV. (*b*).

Suppose no angle in the following list is greater than $+ 180°$.

1. If $\sin A = \frac{3}{5}$, find $\tan A$ and $\operatorname{cosec} A$.

2. If $\cos B = \frac{1}{3}$, find $\sin B$ and $\cot B$.

3. If $\tan A = \frac{4}{3}$, find $\sin A$ and $\sec A$.

4. If $\sec \theta = 4$, find $\cot \theta$ and $\sin \theta$.

5. If $\tan \theta = \sqrt{3}$, find $\sin \theta$ and $\cos \theta$.

6. If $\cot \theta = \dfrac{2}{\sqrt{5}}$, find $\sin \theta$ and $\sec \theta$.

7. If $\sin \theta = \dfrac{b}{c}$, find $\tan \theta$.

8. If $\tan \theta = a$, find $\sin \theta$ and $\cos \theta$.

9. If $\sec \theta = a$, find $\sin \theta$ and $\cot \theta$.

10. If $\sin \theta = a$, and $\tan \theta = b$, prove that $(1 - a^2)(1 + b^2) = 1$.

11. If $\cos \theta = h$, and $\tan \theta = k$, find the equation connecting h and k.

12. If $A < 90°$, and $\tan A + \sec A = 2$, prove that $\sin A = \frac{3}{5}$. Find the remaining ratios of A.

13. Show, using the formulæ Art. 67, that the numerical value of

$\sin A \not> 1$. $\cos A \not> 1$. $\sec A \not< 1$. $\operatorname{cosec} A \not< 1$.

CHAPTER IX

THE SOLUTION OF TRIGONOMETRICAL EQUATIONS

73. The formulæ developed in Chapter VII. and the expressions to be proved in Examples XIV. are true for *any* angle.

E.g. $\sin^2 A + \cos^2 A = 1$, whatever be the value of A. $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ is true whatever value θ may have.

We shall now consider expressions which are true only for certain values of the angle.

E.g. $\sin\theta = \dfrac{\sqrt{2}}{2}$ is an equation satisfied by 45° and 135°, and by no other positive angles less than 360°.

The former expressions are called **Trigonometric Identities.**

The latter expressions are called **Trigonometric Equations.**

The **solution** of a trigonometric equation is the process of finding an angle which, if substituted in the equation, satisfies it.

74. Example 1. Solve the equation

$$\sin\theta - \csc\theta + \tfrac{3}{2} = 0.$$

Substitute in the equation $\csc\theta = \dfrac{1}{\sin\theta}$.

Then $$2\sin^2\theta + 3\sin\theta - 2 = 0,$$

or $$(\sin\theta + 2)(2\sin\theta - 1) = 0.$$

$$\therefore\ \sin\theta = -2, \qquad (1)$$

$$\sin\theta = \tfrac{1}{2}. \qquad (2)$$

No angle satisfies (1). [Example 13, XV., (*b*).]

I. Let us suppose that $\theta < 180°$.

Then from (2) $\theta = 30°$, (Art. 41.)

or, $\theta = 150°$. (Art. 58, Cor.)

II. Suppose θ is unrestricted in magnitude.

Then $\theta = 30°,\ 2\pi + 30°,\ 390°, \cdots 2n\pi + 30°$; (Art. 58, Cor.)

or, $\theta = 150°,\ 2\pi + 150° \cdots 2n\pi + 150°$,

or, in a single statement,

$$\theta = 2n\pi + 30°, \text{ or } (2n+1)\pi - 30°.$$

Another series of angles will satisfy equation (2),

viz. $-330°$, *i.e.* $-2\pi + 30°$, and in general $-2n\pi + 30°$, (Art. 58.)

and $-210°$, *i.e.* $-\pi - 30°$, and in general $-(2n+1)\pi - 30°$, (Art. 58.)

n being a *positive* integer.

Another statement combines the results of II. thus,

$$\theta = 2n\pi + 30°,$$

or

$$\theta = (2n+1)\pi - 30°,$$

n being any integer, *positive* or *negative*, or zero.

EXAMPLE 2. What positive angles satisfy the equation

$$\sqrt{2}\cos\theta = \cot\theta\,?$$

Substitute $$\cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Then $$\sqrt{2}\cos\theta - \frac{\cos\theta}{\sin\theta} = 0,$$

$$\cos\theta\left(\sqrt{2} - \frac{1}{\sin\theta}\right) = 0.$$

$$\therefore\ \cos\theta = 0, \qquad (1)$$

$$\sin\theta = \frac{1}{\sqrt{2}}. \qquad (2)$$

From (1) $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, $2\pi + \frac{\pi}{2}$, $3\pi + \frac{\pi}{2}$, and in general $n\pi + \frac{\pi}{2}$.

From (2) $\theta = 2n\pi + \frac{\pi}{4}$ or $(2n+1)\pi - \frac{\pi}{4}$.

n being a *positive* integer or zero.

EXAMPLES. XVI.

Find all positive angles not greater than 360° satisfying the equations:

1. $\sin\theta = \frac{1}{\sqrt{2}}$.
2. $4\sin\theta = \csc\theta$.
3. $2\cos\theta = \sec\theta$.
4. $4\sin\theta - 3\csc\theta = 0$.
5. $4\cos\theta - 3\sec\theta = 0$.
6. $3\tan\theta = \cot\theta$.
7. $3\sin\theta - 2\cos^2\theta = 0$.
8. $\sqrt{2}\sin\theta = \tan\theta$.
9. $2\cos\theta = \sqrt{3}\cot\theta$.
10. $\tan\theta = 3\cot\theta$.
11. $\tan\theta + \cot\theta = 2$.
12. $2\sin^2\theta + \sqrt{2}\cos\theta = 2$.
13. $2\cos^2\theta + \sqrt{2}\sin\theta = 2$.
14. $3\tan^2\theta - 4\sin^2\theta = 1$.
15. $2\sin^2\theta + \sqrt{2}\sin\theta = 2$.
16. $\cos^2\theta - \sqrt{3}\cos\theta + \frac{3}{4} = 0$.
17. $\cos^2\theta + 2\sin^2\theta - \frac{5}{2}\sin\theta = 0$.

Find all angles, either positive or negative, whose magnitude is not greater than 360°, which satisfy the following equations:

18. $5\tan^2\theta - \sec^2\theta = 11$.
19. $\frac{\sin(180° - \theta)}{2} + \tan\theta = 0$.

Find all angles that satisfy the following equations:

20. $\tan^2\theta = 1$.
21. $\sqrt{3}\sin\theta + 2\cos^2\theta = 2$.
22. $2\sin\theta\cos\theta = 1$.

CHAPTER X

ON THE TRIGONOMETRICAL RATIOS OF TWO OR MORE ANGLES

75. We will now establish the following fundamental formulæ:

$$\left.\begin{aligned}\sin(A+B) &= \sin A \cdot \cos B + \cos A \cdot \sin B\\ \cos(A+B) &= \cos A \cdot \cos B - \sin A \cdot \sin B\\ \sin(A-B) &= \sin A \cdot \cos B - \cos A \cdot \sin B\\ \cos(A-B) &= \cos A \cdot \cos B + \sin A \cdot \sin B\end{aligned}\right\} \qquad \text{(i.)}.$$

Here A and B are angles; so that $(A+B)$ and $(A-B)$ are also angles.

Hence, $\sin(A+B)$ is the sine of an angle, and must not be confounded with $\sin A + \sin B$.

Sin $(A+B)$ is a single fraction.

Sin $A + \sin B$ is the sum of two fractions.

The student should notice that the words of the two proofs of Arts. 76, 77 are very nearly the same.

76. *To prove that*

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B,$$

and that

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B.$$

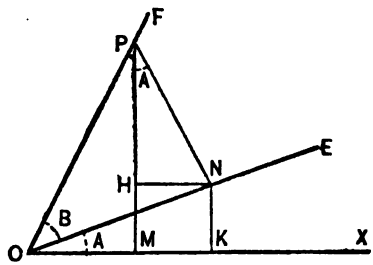


Fig. 46.

Let XOE be the angle A, and EOF the angle B, then XOF is the angle $(A+B)$.

In OF, *the line which is one of the sides of the angle* $(A+B)$, take any point P, and from P draw PM and PN at right angles to OX and OE respectively. Draw NH and NK at right angles to MP and OX respectively. Then the angle

$$HPN = 90° - PNH = HNO = XOE = A.^1$$

Now $$\sin(A+B) = \sin XOF = \frac{MP}{OP} = \frac{MH+HP}{OP} = \frac{KN}{OP} + \frac{HP}{OP}$$

$$= \frac{KN \cdot ON}{ON \cdot OP} + \frac{HP \cdot NP}{NP \cdot OP} = \frac{KN}{ON} \cdot \frac{ON}{OP} + \frac{HP}{NP} \cdot \frac{NP}{OP}$$

$$= \sin XOE \cdot \cos EOF + \cos HPN \cdot \sin EOF$$

$$= \sin A \cdot \cos B + \cos A \cdot \sin B.$$

Also, $$\cos(A+B) = \cos XOF = \frac{OM}{OP} = \frac{OK - MK}{OP} = \frac{OK}{OP} - \frac{HN}{OP}$$

$$= \frac{OK \cdot ON}{ON \cdot OP} - \frac{HN \cdot NP}{NP \cdot OP} = \frac{OK}{ON} \cdot \frac{ON}{OP} - \frac{HN}{NP} \cdot \frac{NP}{OP}$$

$$= \cos XOE \cdot \cos EOF - \sin HPN \cdot \sin EOF$$

$$= \cos A \cdot \cos B - \sin A \cdot \sin B.$$

77. *To prove that*

$$\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B,$$

and that $$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B.$$

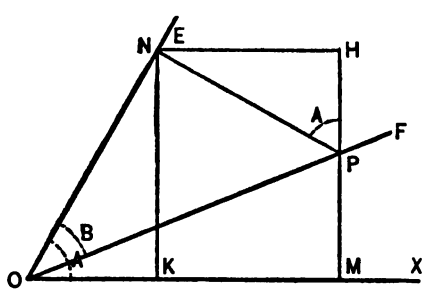


FIG. 47.

[1] Or thus. On OP as diameter, describe a circle. This will pass through M and N, because the angles OMP and ONP are right angles; therefore MPN and MON are angles in the same segment; so that the angle $MPN = MON = A$.

Let XOE be the angle A, and FOE the angle B. Then in the figure, XOF is the angle $(A-B)$.

In OF, *the line which bounds the compound angle* $(A-B)$, take any point P, and from P draw PM, PN at right angles to OX and OE respectively. Draw NH, NK at right angles to MP and OX respectively. Then the angle

$$NPH = 90^\circ - HNP = HNE = XOE = A.^1$$

Now $\sin(A-B) = \sin XOF = \dfrac{MP}{OP} = \dfrac{MH-PH}{OP} = \dfrac{KN}{OP} - \dfrac{PH}{OP}$

$$= \frac{KN \cdot ON}{ON \cdot OP} - \frac{PH \cdot NP}{NP \cdot OP} = \frac{KN}{ON} \cdot \frac{ON}{OP} - \frac{PH}{NP} \cdot \frac{NP}{OP}$$

$$= \sin XOE \cdot \cos FOE - \cos HPN \cdot \sin FOE$$

$$= \sin A \cdot \cos B - \cos A \cdot \sin B.$$

Also, $\cos(A-B) = \cos XOF = \dfrac{OM}{OP} = \dfrac{OK+KM}{OP} = \dfrac{OK}{OP} + \dfrac{NH}{OP}$

$$= \frac{OK \cdot ON}{ON \cdot OP} + \frac{NH \cdot NP}{NP \cdot OP} = \frac{OK}{ON} \cdot \frac{ON}{OP} + \frac{NH}{NP} \cdot \frac{NP}{OP}$$

$$= \cos XOE \cdot \cos FOE + \sin HPN \cdot \sin FOE$$

$$= \cos A \cdot \cos B + \sin A \cdot \sin B.$$

Example. Find the value of $\sin 75^\circ$.

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ \cdot \cos 30^\circ + \cos 45^\circ \cdot \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}.\end{aligned}$$

EXAMPLES. XVII.

1. Show that $\cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.
2. Show that $\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.
3. Show that $\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.
4. Show that $\tan 75^\circ = 2 + \sqrt{3}$.

[1] Or thus. On OP as diameter, describe a circle. This will pass through M and N, because the angles OMP and ONP are right angles; therefore the angles MPN and MON together make up two right angles, so that the angle $HPN = MON = A$.

5. If $\sin A = \frac{4}{5}$ and $\sin B = \frac{3}{5}$, find a value for $\sin(A + B)$ and for $\cos(A - B)$.

6. If $\sin A = .6$ and $\sin B = \frac{5}{13}$, find a value for $\sin(A + B)$ and for $\cos(A + B)$.

7. When $\sin A = \frac{1}{\sqrt{5}}$ and $\sin B = \frac{1}{\sqrt{10}}$, then one value of $(A + B)$ is $45°$.

8. Prove that $\sin 75° = .9659 \cdots$.

9. Prove that $\sin 15° = .2588 \cdots$.

10. Prove that $\tan 15° = .2679 \cdots$.

11. Calculate $\sin 90°$ and $\cos 90°$, using the ratios of $45°$.

***78.** The proofs given in Arts. 76 and 77 are really, at least so far as the figures are concerned, rigorous only in case $(A + B) < 90°$, and hence A and B are each less than $90°$. By a careful regard as to sign of angles and lines, however, the wording will hold for angles of any magnitude. The student may satisfy himself that this is true by constructing suitable figures. The accompanying figure will serve in case $A + B < 180°$ and $A < 90°$, $B > 90°$.

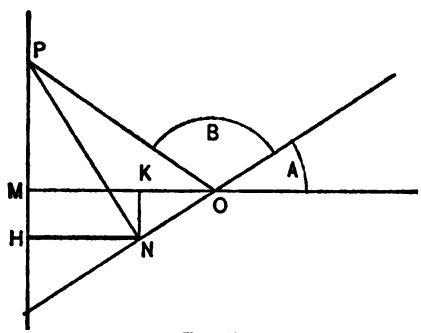


FIG. 48.

***79.** In order, however, to remove the restrictions as to magnitude, placed on A and B, we shall pursue the following course:

Suppose $\quad 180° > A > 90°$ and $B < 90°$.

Put $\quad A = 90° + A'$; then $A' < 90°$.

$$\sin(A + B) = \sin(90° + A' + B) = \sin[90° + (A' + B)], \quad \text{(Art. 64.)}$$

$$\sin[90° + (A' + B)] = \cos(A' + B). \quad \text{(Art. 62.)}$$

A' and B are each less than $90°$. We may therefore write,

$$\cos(A' + B) = \cos A' \cos B - \sin A' \sin B; \quad \text{(Art. 76.)}$$

or

$$\cos(A - 90° + B)$$
$$= \cos(A - 90°)\cos B - \sin(A - 90°)\sin B;$$

since $\cos(A - 90° + B) = \cos[90° - (A + B)]$ (Art. 58 Cor.)

$= \sin(A + B)$; (Art. 61.)

and since $\sin(A - 90°) = -\sin(90° - A)$ (Art. 58.)

$= -\cos A$, (Art. 61.)

we may write,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

By a repetition of this process A may become an angle of any quadrant, and it is evident the reasoning and wording remain just as above; so that the *magnitude of A is unrestricted.*

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B, \qquad \text{(i.)}$$

whatever be the *magnitude of A, provided* $B < 90°$.

There remains yet to inquire what happens if A and B are both greater than 90°.

Suppose $180° > B > 90°$.

Put $B = 90° + B'$; then $B' < 90°$;

$$\sin(A + B) = \sin(90° + A + B') = \cos(A + B').$$

We can now apply (i.), whence

$$\sin(A + B) = \cos A \cos B' - \sin A \sin B'$$
$$= \cos A \sin B + \sin A \cos B.$$

Equation (1) is therefore true if $B > 90°$ but $< 180°$. As before we need only repeat the process in order to remove all restrictions as to the magnitude of B. Hence we conclude that whatever the values of A and B,

$$\sin(A + B) = \sin A \cos B + \sin B \cos A.$$

By a similar process, the three kindred formulæ may be shown to be true for all angles.

NOTE. See Nixon's *Elementary Trigonometry*, Sec. 18.

For an elegant proof of this theorem by the method of projections, see Hobson's *Plane Trigonometry*, Sec. 40.

80. It is important that the student should become thoroughly familiar with the formulæ of Art. 75, and that he should be able to work examples involving their use.

EXAMPLES. XVIII.

Prove the following statements:

1. $\sin(A+B)+\sin(A-B)=2\sin A\cdot\cos B.$
2. $\sin(A+B)-\sin(A-B)=2\cos A\cdot\sin B.$
3. $\cos(A+B)+\cos(A-B)=2\cos A\cdot\cos B.$
4. $\cos(A-B)-\cos(A+B)=2\sin A\cdot\sin B.$
5. $\dfrac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}=\tan A.$
6. $\tan\alpha+\tan\beta=\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cdot\cos\beta}.$
7. $\tan\alpha-\tan\beta=\dfrac{\sin(\alpha-\beta)}{\cos\alpha\cdot\cos\beta}.$
8. $\cot\alpha+\tan\beta=\dfrac{\cos(\alpha-\beta)}{\sin\alpha\cdot\cos\beta}.$
9. $\cot\alpha-\tan\beta=\dfrac{\cos(\alpha+\beta)}{\sin\alpha\cdot\cos\beta}.$
10. $\tan\alpha+\cot\beta=\dfrac{\cos(\alpha-\beta)}{\cos\alpha\cdot\sin\beta}.$
11. $\dfrac{\tan\theta+\tan\phi}{\tan\theta-\tan\phi}=\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$
12. $\dfrac{\tan\theta\cdot\tan\phi+1}{1-\tan\theta\cdot\tan\phi}=\dfrac{\cos(\theta-\phi)}{\cos(\theta+\phi)}.$
13. $\dfrac{\tan\theta+\cot\phi}{\cot\phi-\tan\theta}=\cos(\theta-\phi)\cdot\sec(\theta+\phi).$
14. $\dfrac{\cot\theta+\cot\phi}{\cot\theta-\cot\phi}=-\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$
15. $\dfrac{\tan\theta\cdot\cot\phi+1}{\tan\theta\cdot\cot\phi-1}=\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$
16. $\dfrac{1+\cot\gamma\cdot\tan\delta}{\cot\gamma-\tan\delta}=\tan(\gamma+\delta).$
17. $\dfrac{1-\cot\gamma\cdot\tan\delta}{\cot\gamma+\tan\delta}=\tan(\gamma-\delta).$
18. $\dfrac{\tan\gamma\cdot\cot\delta-1}{\tan\gamma+\cot\delta}=\tan(\gamma-\delta).$
19. $\dfrac{\tan\gamma\cdot\cot\delta+1}{\cot\delta-\tan\gamma}=\tan(\gamma+\delta).$
20. $\dfrac{\cot\delta-\cot\gamma}{\cot\gamma\cdot\cot\delta+1}=\tan(\gamma-\delta).$
21. $\tan^2\alpha-\tan^2\beta=\dfrac{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)}{\cos^2\alpha\cdot\cos^2\beta}.$
22. $\cot^2\alpha-\tan^2\beta=\dfrac{\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)}{\sin^2\alpha\cdot\cos^2\beta}.$
23. $\dfrac{\tan^2\alpha-\tan^2\beta}{1-\tan^2\alpha\cdot\tan^2\beta}=\tan(\alpha+\beta)\cdot\tan(\alpha-\beta).$
24. $\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta=\cos^2\beta-\cos^2\alpha.$
25. $\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha.$
26. $\sin(A-45^\circ)=\dfrac{\sin A-\cos A}{\sqrt{2}}.$
27. $\sqrt{2}\cdot\sin(A+45^\circ)=\sin A+\cos A.$
28. $\cos A-\sin A=\sqrt{2}\cdot\cos(A+45^\circ).$
29. $\cos(A+45^\circ)+\sin(A-45^\circ)=0.$
30. $\cos(A-45^\circ)=\sin(A+45^\circ).$
31. $\sin(\theta+\phi)\cdot\cos\theta-\cos(\theta+\phi)\cdot\sin\theta=\sin\phi.$
32. $\sin(\theta-\phi)\cdot\cos\phi+\cos(\theta-\phi)\cdot\sin\phi=\sin\theta.$
33. $\cos(\theta+\phi)\cdot\cos\theta+\sin(\theta+\phi)\cdot\sin\theta=\cos\phi.$
34. $\dfrac{\tan(\theta-\phi)+\tan\phi}{1-\tan(\theta-\phi)\cdot\tan\phi}=\tan\theta.$

35. $\dfrac{\tan(\theta+\phi)-\tan\theta}{1+\tan(\theta+\phi)\cdot\tan\theta}=\tan\phi.$

36. $2\sin\left(\alpha+\dfrac{\pi}{4}\right)\cdot\cos\left(\beta-\dfrac{\pi}{4}\right)=\cos(\alpha-\beta)+\sin(\alpha+\beta).$

37. $2\sin\left(\dfrac{\pi}{4}-\alpha\right)\cdot\cos\left(\dfrac{\pi}{4}+\beta\right)=\cos(\alpha-\beta)-\sin(\alpha+\beta).$

38. $\cos(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\left(\dfrac{\pi}{4}+\alpha\right)\cdot\cos\left(\dfrac{\pi}{4}+\beta\right).$

39. $\cos(\alpha+\beta)-\sin(\alpha-\beta)=2\sin\left(\dfrac{\pi}{4}-\alpha\right)\cdot\cos\left(\dfrac{\pi}{4}-\beta\right).$

40. $\sin nA\cdot\cos A+\cos nA\cdot\sin A=\sin(n+1)A.$

41. $\cos(n-1)A\cdot\cos A-\sin(n-1)A\cdot\sin A=\cos nA.$

42. $\sin nA\cdot\cos(n-1)A-\cos nA\cdot\sin(n-1)A=\sin A.$

43. $\cos(n-1)A\cdot\cos(n+1)A-\sin(n-1)A\cdot\sin(n+1)A=\cos 2nA.$

44. $(\cos A+\sin A)(\cos B+\sin B)=\cos(A-B)+\sin(A+B).$

81. The following formulæ are important:

$$\left.\begin{aligned}\tan(A+B)&=\frac{\tan A+\tan B}{1-\tan A\cdot\tan B};\\ \tan(A-B)&=\frac{\tan A-\tan B}{1+\tan A\cdot\tan B}.\end{aligned}\right\}\qquad\text{(ii.)}$$

The proof of the first is given below. The student may prove the second in a similar manner. Or, since these formulæ are true for all values of A and B, we may substitute $-B$, for B; the formula now becomes $\tan(A-B)=\dfrac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}=\dfrac{\tan A-\tan B}{1+\tan A\tan B}.$

Example. To prove $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}.$

(i.) By using the results of Arts. 76, 77, we have

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cdot\cos B+\cos A\cdot\sin B}{\cos A\cdot\cos B-\sin A\cdot\sin B}.$$

Divide the numerator and the denominator of this fraction each by $\cos A\cdot\cos B$, and we get

$$\tan(A+B)=\frac{\dfrac{\sin A\cdot\cos B}{\cos A\cdot\cos B}+\dfrac{\cos A\cdot\sin B}{\cos A\cdot\cos B}}{\dfrac{\cos A\cdot\cos B}{\cos A\cdot\cos B}-\dfrac{\sin A\cdot\sin B}{\cos A\cdot\cos B}}=\frac{\tan A+\tan B}{1-\tan A\cdot\tan B}.\qquad\text{Q.E.D.}$$

EXAMPLES. XIX.

1. If $\tan A=\frac{1}{2}$ and $\tan B=\frac{1}{4}$, prove that $\tan(A+B)=\frac{6}{7}$, and $\tan(A-B)=\frac{2}{9}$.

2. If $\tan A=1$ and $\tan B=\dfrac{1}{\sqrt{3}}$, prove that $\tan(A+B)=2+\sqrt{3}$.

3. Prove that $\tan 15^\circ=2-\sqrt{3}$.

4. If $\tan A = \frac{5}{6}$ and $\tan B = \frac{1}{11}$, prove that $\tan(A+B) = 1$. What is $(A+B)$ in this case?

5. If $\tan A = m$ and $\tan B = \frac{1}{m}$, prove that $\tan(A+B) = \infty$. What is $(A+B)$ in this case?

Prove the following statements:

6. $\cot(A+B) = \dfrac{\cot A \cdot \cot B - 1}{\cot A + \cot B}$.

7. $\cot(A-B) = \dfrac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$.

8. $\cot\left(\theta - \dfrac{\pi}{4}\right) = \dfrac{\cot\theta + 1}{1 - \cot\theta}$.

9. $\dfrac{\cot\theta - 1}{\cot\theta + 1} = \cot\left(\theta + \dfrac{\pi}{4}\right)$.

10. $\tan\left(\theta - \dfrac{\pi}{4}\right) + \cot\left(\theta + \dfrac{\pi}{4}\right) = 0$.

11. $\cot\left(\theta - \dfrac{\pi}{4}\right) + \tan\left(\theta + \dfrac{\pi}{4}\right) = 0$.

12. If $\tan\alpha = \dfrac{m}{m+1}$ and $\tan\beta = \dfrac{1}{2m+1}$, prove that $\tan(\alpha+\beta) = 1$.

13. $\dfrac{\tan(n+1)\phi - \tan n\phi}{1 + \tan(n+1)\phi \cdot \tan n\phi} = \tan\phi$.

14. $\dfrac{\tan(n+1)\phi + \tan(1-n)\phi}{1 - \tan(n+1)\phi \cdot \tan(1-n)\phi} = \tan 2\phi$.

15. If $\tan\alpha = m$ and $\tan\beta = n$, prove that

$$\cos(\alpha+\beta) = \frac{1 - mn}{\sqrt{(1+m^2)(1+n^2)}}.$$

16. If $\tan\alpha = (a+1)$ and $\tan\beta = (a-1)$, then $2\cot(\alpha-\beta) = a^2$.

17. If $\alpha + \beta + \gamma = 90°$, then $\tan\gamma = \dfrac{1 - \tan\alpha\tan\beta}{\tan\alpha + \tan\beta}$.

82. From Art. 75 we have

$$\left.\begin{aligned}
\sin(A+B) &= \sin A \cdot \cos B + \cos A \cdot \sin B;\\
\sin(A-B) &= \sin A \cdot \cos B - \cos A \cdot \sin B;\\
\cos(A+B) &= \cos A \cdot \cos B - \sin A \cdot \sin B;\\
\cos(A-B) &= \cos A \cdot \cos B + \sin A \cdot \sin B.
\end{aligned}\right\} \qquad \text{(i.)}$$

From these by addition and subtraction we get

$$\left.\begin{aligned}
\sin(A+B) + \sin(A-B) &= 2\sin A \cdot \cos B;\\
\sin(A+B) - \sin(A-B) &= 2\cos A \cdot \sin B;\\
\cos(A+B) + \cos(A-B) &= 2\cos A \cdot \cos B;\\
\cos(A-B) - \cos(A+B) &= 2\sin A \cdot \sin B.
\end{aligned}\right\} \qquad \text{(ii.)}$$

Now put S for $(A+B)$, and put T for $(A-B)$:

Then $\qquad S + T = 2A$, and $S - T = 2B$,

so that $\qquad A = \dfrac{S+T}{2}$, and $B = \dfrac{S-T}{2}$.

Hence the above results may be written

$$\left.\begin{aligned}
\sin S + \sin T &= 2\sin\frac{S+T}{2}\cdot\cos\frac{S-T}{2};\\
\sin S - \sin T &= 2\cos\frac{S+T}{2}\cdot\sin\frac{S-T}{2};\\
\cos S + \cos T &= 2\cos\frac{S+T}{2}\cdot\cos\frac{S-T}{2};\\
{}^{1}\cos T - \cos S &= 2\sin\frac{S+T}{2}\cdot\sin\frac{S-T}{2}.
\end{aligned}\right\}\qquad\text{(iii.)}$$

83. The formulæ (iii.) are most important, and the student is recommended to get thoroughly familiar with them in *words*, as follows:

(1) *The sum of the sines of two angles equals twice the sine of half their sum multiplied by the cosine of half their difference.*

(2) *The difference of the sines of two angles equals twice the cosine of half their sum multiplied by the sine of half their difference.*

(3) *The sum of the cosines of two angles equals twice the cosine of half their sum multiplied by the cosine of half their difference.*

(4) *The difference of the cosines*[2] *of two angles equals twice the sine of half their sum multiplied by the sine of half their difference.*

84. It will be convenient to refer to the formulæ (i.) as the 'A, B' formulæ, and to the formulæ (iii.) as the 'S, T' formulæ.

EXAMPLES. XX.

Prove the following statements:

1. $\sin 60^\circ + \sin 30^\circ = 2\sin 45^\circ\cdot\cos 15^\circ.$
2. $\sin 60^\circ + \sin 20^\circ = 2\sin 40^\circ\cdot\cos 20^\circ.$
3. $\sin 40^\circ - \sin 10^\circ = 2\cos 25^\circ\cdot\sin 15^\circ.$
4. $\cos\frac{\pi}{3} + \cos\frac{\pi}{2} = 2\cos\frac{5\pi}{12}\cdot\cos\frac{\pi}{12}.$
5. $\cos\frac{\pi}{3} - \cos\frac{\pi}{2} = 2\sin\frac{5\pi}{12}\cdot\sin\frac{\pi}{12}.$
6. $\sin 3A + \sin 5A = 2\sin 4A\cdot\cos A.$
7. $\sin 7A - \sin 5A = 2\cos 6A\cdot\sin A.$
8. $\cos 5A + \cos 9A = 2\cos 7A\cdot\cos 2A.$

[1] If A and B are each less than 90°, then S, which is their *sum*, is greater than T, their *difference*. Therefore if S be less than 90°, $\cos S$ is less than $\cos T$; so that $\cos T - \cos S$ is *positive*.

[2] The difference of the cosines of two angles is the cosine of the smaller angle – the cosine of the greater angle.

9. $\cos 5A - \cos 4A = -2 \sin \frac{9A}{2} \cdot \sin \frac{A}{2}$.

10. $\cos A - \cos 2A = 2 \sin \frac{3A}{2} \cdot \sin \frac{A}{2}$.

11. $\frac{\sin 2\theta + \sin \theta}{\cos \theta + \cos 2\theta} = \tan \frac{3\theta}{2}$.

12. $\frac{\sin 2\theta - \sin \theta}{\cos \theta - \cos 2\theta} = \cot \frac{3\theta}{2}$.

13. $\frac{\sin 3\theta + \sin 2\theta}{\cos 2\theta - \cos 3\theta} = \cot \frac{\theta}{2}$.

14. $\frac{\sin \theta + \sin \phi}{\cos \theta - \cos \phi} = \frac{\cos \theta + \cos \phi}{\sin \phi - \sin \theta}$.

15. $\cos (60° + A) + \cos (60° - A) = \cos A$.

16. $\cos (45° + A) + \cos (45° - A) = \sqrt{2} \cdot \cos A$.

17. $\sin (45° + A) - \sin (45° - A) = \sqrt{2} \cdot \sin A$.

18. $\cos (30° - A) - \cos (30° + A) = \sin A$.

19. $\frac{\sin \theta - \sin \phi}{\cos \phi - \cos \theta} = \cot \frac{\theta + \phi}{2}$.

20. $\frac{\sin \theta - \sin \phi}{\sin \theta + \sin \phi} = \cot \left(\frac{\theta + \phi}{2}\right) \cdot \tan \left(\frac{\theta - \phi}{2}\right)$.

85. It is important that the student should be thoroughly familiar with the second set of formulæ on Art. 82.

Written as follows, they may be regarded as the inverse of the 'S, T' formulæ.

$$\left.\begin{aligned} 2 \sin A \cdot \cos B &= \sin (A + B) + \sin (A - B); \\ 2 \cos A \cdot \sin B &= \sin (A + B) - \sin (A - B); \\ 2 \cos A \cdot \cos B &= \cos (A + B) + \cos (A - B); \\ 2 \sin A \cdot \sin B &= \cos (A - B) - \cos (A + B). \end{aligned}\right\} \qquad \text{(iv.)}$$

EXAMPLES. XXI.

Express as the sum or as the difference of two trigonometrical ratios the ten following expressions:

1. $2 \sin \theta \cdot \cos \phi$.
2. $2 \cos \alpha \cdot \cos \beta$.
3. $2 \sin 2\alpha \cdot \cos 3\beta$.
4. $2 \cos (\alpha + \beta) \cdot \cos (\alpha - \beta)$.
5. $2 \sin 3\theta \cdot \cos 5\theta$.
6. $2 \cos \frac{3\theta}{2} \cdot \cos \frac{\theta}{2}$.
7. $\sin 4\theta \cdot \sin \theta$.
8. $\cos \frac{5\theta}{2} \cdot \sin \frac{3\theta}{2}$.
9. $2 \cos 10° \cdot \sin 50°$.
10. $\cos 45° \cdot \sin 15°$.
11. Simplify $2 \cos 2\theta \cdot \cos \theta - 2 \sin 4\theta \cdot \sin \theta$.
12. Simplify $\sin \frac{5\theta}{2} \cdot \cos \frac{\theta}{2} - \sin \frac{9\theta}{2} \cdot \cos \frac{3\theta}{2}$.
13. Simplify $\sin 3\theta + \sin 2\theta + 2 \sin \frac{3\theta}{2} \cdot \cos \frac{\theta}{2}$.
14. Prove that $\sin \frac{11\theta}{4} \cdot \sin \frac{\theta}{4} + \sin \frac{7\theta}{4} \cdot \sin \frac{3\theta}{4} = \sin 2\theta \cdot \sin \theta$.

***86.** Since A and B are any angles, we may substitute for A, $\alpha + \beta$, and for B, γ; then

$$\begin{aligned}\sin(A+B) &= \sin(\alpha+\beta+\gamma)\\ &= \sin(\alpha+\beta)\cos\gamma + \cos(\alpha+\beta)\sin\gamma\\ &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)\cos\gamma\\ &\quad + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\sin\gamma.\end{aligned}$$

$$\begin{aligned}\therefore\ \sin(\alpha+\beta+\gamma) &= \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\alpha\cos\gamma\\ &\quad + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma. \qquad (1)\end{aligned}$$

It is evident that these formulæ may be extended in this way to include any number of angles that we choose.

EXAMPLES. XXII.

Find the expressions similar to (1) for

1. $\cos(\alpha+\beta+\gamma)$.
2. $\sin(\alpha+\beta-\gamma)$.
3. $\cos(\alpha+\beta-\gamma)$.
4. $\sin(\alpha\pm\beta\pm\gamma)$.

***87.** The formulæ of Arts. 86 and 75 are called the addition formulæ of the Trigonometric functions.

CHAPTER XI

ON THE TRIGONOMETRICAL RATIOS OF MULTIPLE ANGLES AND SUBMULTIPLE ANGLES

88. To express the **trigonometrical ratios** of the **angle $2A$** in terms of those of the angle A.

Since $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$;

$$\therefore \sin(A+A) = \sin A \cdot \cos A + \cos A \cdot \sin A;$$

$$\therefore \sin 2A = 2 \sin A \cdot \cos A. \tag{1}$$

Also, since $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$;

$$\therefore \cos(A+A) = \cos A \cdot \cos A - \sin A \cdot \sin A;$$

$$\therefore \cos 2A = \cos^2 A - \sin^2 A. \tag{2}$$

But $$1 = \cos^2 A + \sin^2 A;$$

$$\therefore 1 + \cos 2A = 2\cos^2 A,$$

$$\text{and } 1 - \cos 2A = 2\sin^2 A.$$

The last two results are usually written

$$\cos 2A = 2\cos^2 A - 1, \tag{3}$$

$$\text{and } \cos 2A = 1 - 2\sin^2 A. \tag{4}$$

Again, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$;

$$\therefore \tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \cdot \tan A};$$

$$\therefore \tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \tag{5}$$

89. These five formulæ are very important,

$$\left.\begin{array}{ll} \sin 2A = 2\sin A \cdot \cos A; & (1) \\ \left.\begin{array}{ll} \cos 2A = \cos^2 A - \sin^2 A; & (2) \\ \cos 2A = 2\cos^2 A - 1; & (3) \\ \cos 2A = 1 - 2\sin^2 A; & (4) \end{array}\right\} & \\ \tan 2A = \dfrac{2\tan A}{1-\tan^2 A}. & (5) \end{array}\right\} \text{(v.)}$$

90. The following result is important,

$$\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A \cdot \cos A}{2\cos^2 A} = \tan A.$$

91. Submultiple Angles. In formulæ (v.) the angle A is *any* angle. Hence we may write $A = \frac{\alpha}{2}$.

The formulæ (v.) now become

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2};$$

$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2};$$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1;$$

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2};$$

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}.$$

***92.** An examination of formulæ (v.) shows that if $\sin A$ or $\cos A$ be given, $\cos 2A$ is uniquely determined. The converse is not true; *i.e.* if $\cos 2A$ is given, $\sin A$ and $\cos A$ have a sign **ambiguity**.

We have $$\cos^2 A = \frac{\cos 2A + 1}{2}.$$

$$\therefore \cos A = \pm\sqrt{\frac{\cos 2A + 1}{2}}.$$

Similarly, $$\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}};$$

$$\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}.$$

If $\sin 2A$ *be given*, we have

$$\sin 2A + 1 = 2\sin A\cos A + \cos^2 A + \sin^2 A.$$

Since $\sin^2 A + \cos^2 A = 1,$

$$\therefore \pm\sqrt{\sin 2A + 1} = \cos A + \sin A.$$

Similarly, $\pm\sqrt{1 - \sin 2A} = \sin A - \cos A,$

whence $$2\sin A = \pm\sqrt{\sin 2A + 1} \pm \sqrt{1 - \sin 2A}.$$

This presents a fourfold sign ambiguity.

Similar remarks apply with equal force to the submultiple angles.

EXAMPLES. XXIII.

Prove the following identities:

1. $2\csc 2A = \sec A \cdot \csc A.$
2. $\dfrac{\operatorname{cosec}^2 A}{\operatorname{cosec}^2 A - 2} = \sec 2A.$
3. $\dfrac{2 - \sec^2 A}{\sec^2 A} = \cos 2A.$
4. $\cos^2 A(1 - \tan^2 A) = \cos 2A.$
5. $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}.$
6. $\dfrac{2\tan B}{1 + \tan^2 B} = \sin 2B.$
7. $\tan B + \cot B = 2\operatorname{cosec} 2B.$
8. $\dfrac{1 - \tan^2 B}{1 + \tan^2 B} = \cos 2B.$
9. $\cot B - \tan B = 2\cot 2B.$
10. $\dfrac{\cot^2 B + 1}{\cot^2 B - 1} = \sec 2B.$
11. $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta.$
12. $\left(\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2}\right)^2 = 1 - \sin\theta.$
13. $\cos^2\dfrac{\theta}{2}\left(1 + \tan\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta.$
14. $\sin^2\dfrac{\theta}{2}\left(\cot\dfrac{\theta}{2} - 1\right)^2 = 1 - \sin\theta.$
15. $\left(\dfrac{\tan\dfrac{\theta}{2} + 1}{\tan\dfrac{\theta}{2} - 1}\right)^2 = \dfrac{1 + \sin\theta}{1 - \sin\theta}.$
16. $\dfrac{\sin\beta}{1 + \cos\beta} = \tan\dfrac{\beta}{2}.$
17. $\dfrac{\sin\beta}{1 - \cos\beta} = \cot\dfrac{\beta}{2}.$
18. $\dfrac{1 - \cos\beta}{1 + \cos\beta} = \tan^2\dfrac{\beta}{2}.$
19. $\dfrac{1 + \sec\beta}{\sec\beta} = 2\cos^2\dfrac{\beta}{2}.$
20. $\operatorname{cosec}\beta - \cot\beta = \tan\dfrac{\beta}{2}.$
21. $\dfrac{\cos 2x}{1 + \sin 2x} = \dfrac{1 - \tan x}{1 + \tan x}.$
22. $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \tan\dfrac{x}{2}}{1 - \tan\dfrac{x}{2}}.$
23. $\dfrac{\cos x}{1 + \sin x} = \dfrac{\cot\dfrac{x}{2} - 1}{\cot\dfrac{x}{2} + 1}.$
24. $\dfrac{\cos x}{1 - \sin x} = \dfrac{\cot\dfrac{x}{2} + 1}{\cot\dfrac{x}{2} - 1}.$
25. $\dfrac{1 + \sin x + \cos x}{1 + \sin x - \cos x} = \cot\dfrac{x}{2}.$
26. $\dfrac{\cos^3\alpha + \sin^3\alpha}{\cos\alpha + \sin\alpha} = \dfrac{2 - \sin 2\alpha}{2}.$
27. $\dfrac{\cos^3\alpha - \sin^3\alpha}{\cos\alpha - \sin\alpha} = \dfrac{2 + \sin 2\alpha}{2}.$
28. $\cos^4\alpha - \sin^4\alpha = \cos 2\alpha.$
29. $\cos^6\alpha + \sin^6\alpha = \dfrac{1 + 3\cos^2 2\alpha}{4}.$

30. $\cos^6 \alpha - \sin^6 \alpha = \dfrac{(3+\cos^2 2\alpha)\cos 2\alpha}{4}$.

31. $\dfrac{\sin 3\beta}{\sin \beta} - \dfrac{\cos 3\beta}{\cos \beta} = 2$.

32. $\dfrac{\cos 3\beta}{\sin \beta} + \dfrac{\sin 3\beta}{\cos \beta} = 2 \cot 2\beta$.

33. $\dfrac{\sin 4\beta}{\sin 2\beta} = 2 \cos 2\beta$.

34. $\dfrac{\sin 5\beta}{\sin \beta} - \dfrac{\cos 5\beta}{\cos \beta} = 4 \cos 2\beta$.

35. $\dfrac{\sin \frac{5\pi}{12}}{\sin \frac{\pi}{12}} - \dfrac{\cos \frac{5\pi}{12}}{\cos \frac{\pi}{12}} = 2\sqrt{3}$.

36. $\tan(45° + A) - \tan(45° - A) = 2 \tan 2A$.

37. $\tan(45° - A) + \cot(45° - A) = 2 \sec 2A$.

38. $\dfrac{\tan^2(45° + A) - 1}{\tan^2(45° + A) + 1} = \sin 2A$.

39. $\dfrac{\sec A + \tan A}{\sec A - \tan A} = \tan\left(45° + \dfrac{A}{2}\right) \cdot \cot\left(45° - \dfrac{A}{2}\right)$.

40. $\dfrac{\cos(A + 45°)}{\cos(A - 45°)} = \sec 2A - \tan 2A$.

41. $\tan B = \dfrac{\sin B + \sin 2B}{1 + \cos B + \cos 2B}$.

42. $\tan B = \dfrac{\sin 2B - \sin B}{1 - \cos B + \cos 2B}$.

93. The following two formulæ should be remembered:

$$\left.\begin{aligned} \sin 3A &= 3 \sin A - 4 \sin^3 A; \\ \cos 3A &= 4 \cos^3 A - 3 \cos A. \end{aligned}\right\} \qquad \text{(vi.)}$$

NOTE. The similarity of these two results is likely to cause confusion. This may be avoided by observing that the second formula must be true when $A = 0°$; and then $\cos 3A = \cos 0° = 1$. In which case the formula gives $\cos 0° = 4 \cos 0° - 3 \cos 0°$, or $1 = 4 - 3$, which is true.

The first formula may be proved thus:

$$\begin{aligned} \sin 3A &= \sin(2A + A) = \sin 2A \cdot \cos A + \cos 2A \cdot \sin A \\ &= (2 \sin A \cdot \cos A) \cos A + (1 - 2 \sin^2 A) \sin A \\ &= 2 \sin A \cdot \cos^2 A + \sin A - 2 \sin^3 A \\ &= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A \\ &= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A \\ &= 3 \sin A - 4 \sin^3 A. \end{aligned}$$

The second formula may be proved in a similar manner.

EXAMPLE. Prove that

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

$$\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \cdot \tan A}$$

$$= \frac{\dfrac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2 \tan A}{1 - \tan^2 A} \cdot \tan A} = \frac{2 \tan A + \tan A - \tan^3 A}{1 - \tan^2 A - 2 \tan^2 A}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

EXAMPLES. XXIV.

Prove the following statements:

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CHAPTER XII

INVERSE TRIGONOMETRIC FUNCTIONS

94. From the equation $y = \sin\phi$, we know that ϕ is an angle whose sine is y. The last statement is expressed by the notation $\phi = \sin^{-1}y$.

Hence $\sin^{-1}y$ is an **angle.**

$\sin^{-1}y$ is sometimes read, "an angle whose sine is y," sometimes "arc-sine y," but more frequently "anti-sine y." But it must be remembered that it means "an angle whose sine is y."

$\cos^{-1}y$ means "an angle whose **cosine** is y."

$\tan^{-1}y$ means "an angle whose **tangent** is y."

$\csc^{-1}y$ means "an angle whose **cosecant** is y."

$\sec^{-1}y$ means "an angle whose **secant** is y."

$\cot^{-1}y$ means "an angle whose **cotangent** is y."

These are read "anti-cosine y," "anti-tangent y," "anti-cosecant y," "anti-secant y," "anti-cotangent y," respectively.

EXAMPLE. $30° = \sin^{-1}\frac{1}{2}$; $45° = \sin^{-1}\frac{1}{\sqrt{2}}$.

95. The expressions $\sin^{-1}y$, $\cos^{-1}y$, etc., are called the *Inverse Trigonometric Functions* or *Inverse Circular Functions.*

96. In Art. 58 we showed that an infinite number of angles, differing by 2π, have the same ratios. Accordingly an infinite number of angles will satisfy an equation of the form $\phi = \sin^{-1}y$, $\theta = \cos^{-1}x$, etc. Accordingly for the sake of definiteness we shall (unless otherwise stated) make the following conventions:

(1) When we are given either of the equations $\phi = \sin^{-1}y$, $\phi = \tan^{-1}y$, $\phi = \csc^{-1}y$, $\phi = \cot^{-1}y$, we shall understand ϕ to be an angle, either *positive* or *negative*, whose magnitude is not greater than 90°.

(2) When we are given the equations $\phi = \cos^{-1}y$, $\phi = \sec^{-1}y$, we shall limit ϕ to a *positive* angle whose magnitude is not greater than 180°.

With these agreements one value and but one will satisfy any of these equations.

If, however, ϕ *is given*, we can always write *a definite equation.*

For example, $225° = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. But if we had given us

$$\phi = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right),$$

and had known nothing else whatever of ϕ, by our agreements we would have concluded $\phi = -45°$.

EXAMPLE 2. Given $\phi = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

By our agreements we know that $\phi = 135°$. We can now write

$$135° = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right).$$

If, on the other hand, we are given

$$\phi = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right), \text{ we conclude } \phi = 45°.$$

97. *To express the inverse ratios in terms of a given one.*

e.g. if $\theta = \tan^{-1}x$, to express the inverse trigonometric functions in terms of x.

$$x = \tan\theta; \qquad \therefore\ \theta = \tan^{-1}x.$$

$$\frac{1}{x} = \cot\theta; \qquad \therefore\ \theta = \cot^{-1}\frac{1}{x}.$$

$$\sqrt{1+x^2} = \sec\theta; \qquad \therefore\ \theta = \sec^{-1}\sqrt{1+x^2}.$$

$$\frac{1}{\sqrt{1+x^2}} = \cos\theta; \qquad \therefore\ \theta = \cos^{-1}\frac{1}{\sqrt{1+x^2}}.$$

$$\frac{x}{\sqrt{1+x^2}} = \sin\theta; \qquad \therefore\ \theta = \sin^{-1}\frac{x}{\sqrt{1+x^2}}.$$

$$\frac{\sqrt{1+x^2}}{x} = \csc\theta; \qquad \therefore\ \theta = \csc^{-1}\frac{\sqrt{1+x^2}}{x}.$$

No ambiguity of sign exists.

For if x is positive, $\sin\theta$ is positive; therefore $\sqrt{1+x^2}$ is positive. If x is negative, $\sin\theta$ is negative; hence $\sqrt{1+x^2}$ is positive. Then by convention $\cos\theta$ is positive; therefore $\sqrt{1+x^2}$ is positive.

EXERCISE. If $x = \cos\theta$, express the inverse trigonometric functions in terms of x.

98. Let $x = \sin\theta$.

Then
$$x = \cos\left(\frac{\pi}{2} - \theta\right);$$
$$\therefore\ \theta = \sin^{-1}x.$$
$$\frac{\pi}{2} - \theta = \cos^{-1}x;$$
$$\therefore\ \theta = \frac{\pi}{2} - \cos^{-1}x;$$
$$\therefore\ \sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x.$$

Exercise. Show
$$\tan^{-1}x = \frac{\pi}{2} - \cot^{-1}x;$$
$$\sec^{-1}x = \frac{\pi}{2} - \csc^{-1}x.$$

99. *To express two inverse trigonometric functions as a single inverse function;*

e.g. consider $\sin^{-1}x + \sin^{-1}y$.

Put
$$x = \sin A;\ \therefore\ A = \sin^{-1}x.$$
$$y = \sin B;\ \therefore\ B = \sin^{-1}y;$$
$$\therefore\ \sin(A + B) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}.$$
$$A + B = \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}).$$

Exercise. Express $\tan^{-1}x + \tan^{-1}y$ as a single inverse trigonometric function.

EXAMPLES. XXV.

1. $\phi = \tan^{-1}(\frac{1}{3})$; find $\cos\phi$.
2. $\phi = \cos^{-1}(-\frac{1}{2})$ find $\csc\phi$.
3. $\phi = \sin^{-1}(-\frac{1}{2})$ find $\sec\phi$; find $\tan\phi$.
4. $\phi = \tan^{-1}\dfrac{m}{m+1}$, $\theta = \tan^{-1}\dfrac{1}{2m+1}$; show that $(\theta + \phi) = \dfrac{\pi}{4}$.
5. $\sin^{-1}\dfrac{3}{5} + \cos^{-1}\dfrac{3}{5} = \dfrac{\pi}{2}$.
6. $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$.
7. $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1 - x^2}$.
8. $\sin(2\sin^{-1}x) = 2x\sqrt{1 - x^2}$.
9. $\sin^{-1}\sqrt{\dfrac{x - a + b}{2b}} = \cos^{-1}\sqrt{\dfrac{b + a - x}{2b}}$.
10. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}(xy \pm \sqrt{(1 - x^2)(1 - y^2)})$.
11. $\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}$.
12. $\tan^{-1}2 + \tan^{-1}\dfrac{1}{2} = \dfrac{\pi}{2}$.

13. Find value of $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\frac{\sqrt{3}}{2}\right)$.

14. $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{16}{65}$.

15. If $\tan^{-1}ky=\tan^{-1}ks-gkx$, then $y=\frac{1}{k}\cdot\frac{2k-\tan gkx}{1+2k\tan kgx}$.

16. $\sin(\sin^{-1}x+\sin^{-1}\sqrt{1-x^2})=1$.

17. $\sin^{-1}\left(\tan\frac{\pi}{4}\right)=2\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

CHAPTER XIII

ON THE RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE

100. The **three sides** and the **three angles** of any **triangle** are called its **six parts.**

By the letters A, B, C we shall indicate

geometrically, the three angular points of the triangle ABC;

algebraically, the three angles at those angular points respectively.

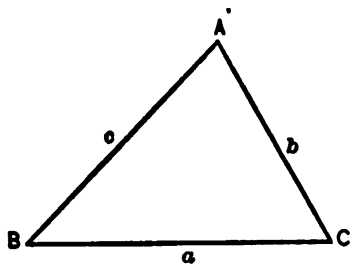


FIG. 49.

By the letters a, b, c, we shall indicate the measures of the sides BC, CA, AB, opposite the angles A, B, C, respectively.

101. I. We know that $A + B + C = 180°$. (Geom.)

102. Also if A be an angle of a triangle, then A may have any value between 0° and 180°. Hence,

(i.) **sin** A must be positive (and less than 1);

(ii.) **cos** A may be positive or negative (but must be numerically less than 1);

(iii.) **tan** A may have any value whatever, positive or negative.

103. Also, if we are given the value of

(i.) **sin** A, there are two angles, each less than 180°, which have the given positive value for their sine.

(ii.) **cos A**, or (iii.) **tan A**, then there is only **one** value of A, which value can be found from the Tables.

104. $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90°$. Therefore $\frac{A}{2}$ is less than 90°, and its Trigonometrical Ratios are all positive. Also, $\frac{A}{2}$ is known, when the value of any one of its ratios is given. Similar remarks of course apply to the angles B and C.

EXAMPLE 1. *To prove* $\sin(A+B)=\sin C$.

$$A+B+C=180°;\ \therefore\ A+B=180°-C,$$

and $$\therefore\ \sin(A+B)=\sin(180°-C)=\sin C. \qquad \text{(Art. 59.)}$$

EXAMPLE 2. *To prove* $\sin\frac{A+B}{2}=\cos\frac{C}{2}$.

Now $$\frac{A+B+C}{2}=90°;\ \therefore\ \frac{A+B}{2}=90°-\frac{C}{2},$$

and $$\therefore\ \sin\frac{A+B}{2}=\sin\left(90°-\frac{C}{2}\right)=\cos\frac{C}{2}. \qquad \text{(Art. 61.)}$$

EXAMPLES. XXVI.

Find A from each of the six following equations, A being an angle of a triangle:

1. $\cos A=\frac{1}{2}$.
2. $\cos A=-\frac{1}{2}$.
3. $\sin A=\frac{1}{2}$.
4. $\tan A=-1$.
5. $\sqrt{2}\sin A=1$.
6. $\tan A=-\sqrt{3}$.

Prove the following statements, A, B, C being the angles of a triangle:

7. $\sin(A+B+C)=0$.
8. $\cos(A+B+C)=-1$.
9. $\sin\frac{1}{2}(A+B+C)=1$.
10. $\cos\frac{1}{2}(A+B+C)=0$.
11. $\tan(A+B)=-\tan C$.
12. $\cot\frac{1}{2}(B+C)=\tan\frac{1}{2}A$.
13. $\cos(A+B)=-\cos C$.
14. $\cos(A+B-C)=-\cos 2C$.
15. $\tan A-\cot B=\cos C\cdot\sec A\cdot\csc B$.
16. $\frac{\sin A-\sin B}{\sin A+\sin B}=\tan\frac{C}{2}\cdot\tan\frac{A-B}{2}$.
17. $\frac{\sin 3B-\sin 3C}{\cos 3C-\cos 3B}=\tan\frac{3A}{2}$.

105. II. *To prove* $a=b\cos C+c\cos B$.

From A, any one of the angular points, draw AD perpendicular to BC, or to BC produced if necessary.

There will be three cases. Fig. i. when both B and C are acute angles; Fig. ii. when one of them (B) is obtuse; Fig. iii. when one of them (B) is a right angle. Then,

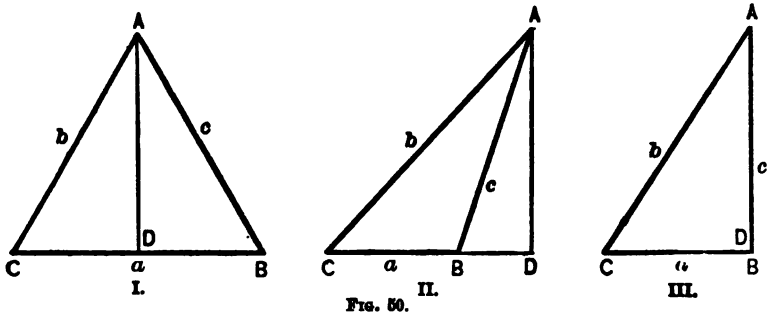


FIG. 50.

Fig. i. $\dfrac{CD}{CA} = \cos ACD$; or, $CD = b \cos C$,

and $\dfrac{DB}{AB} = \cos ABD$; or, $DB = c \cos B$,

$\therefore\ a = CD + DB = b \cos C + c \cos B.$

Fig. ii. $\dfrac{CD}{CA} = \cos ACD$; or, $CD = b \cos C$,

$\dfrac{BD}{AB} = \cos ABD$; or, $BD = c \cos (180° - B)$,

$\therefore\ a = CD - BD = b \cos C - c \cos (180° - B)$

$= b \cos C + c \cos B.$

Fig. iii. $a = CB = b \cos C$

$= b \cos C + c \cos B.$ [For, $\cos B = \cos 90° = 0$.]

Similarly it may be proved that,

$$b = c \cos A + a \cos C;\ c = a \cos B + b \cos A.$$

106. III. *To prove that, in any triangle, the sides are proportional to the sines of the angles opposite;* or, *To prove that*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

From A, any one of the angular points, draw AD perpendicular to BC, or to BC produced if necessary. Then,

I. Fig. 50. $AD = b \sin C$; for, $\dfrac{AD}{AC} = \sin C$ [Def.];

also $AD = c \sin B$; for, $\frac{DA}{AB} = \sin B$.

$$\therefore\ b \sin C = c \sin B;$$

or,
$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

II. Fig. 50. $$AD = b \sin C,$$

and
$$AD = c \sin ABD = c \sin (180° - B).$$

$$\therefore\ AD = c \sin B;$$

$$\therefore\ b \sin C = c \sin B;$$

or,
$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

III. Fig. 50. $AB = AC \cdot \sin C$; or, $c = b \sin C$;

$$\therefore\ \frac{c}{\sin C} = \frac{b}{\sin B}. \quad [\text{For } \sin B = \sin 90° = 1.]$$

Similarly it may be proved that

$$\frac{a}{\sin A} = \frac{b}{\sin B};$$

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad \text{Q.E.D.}$$

107. IV. To prove that $a^2 = b^2 + c^2 - 2\,bc \cos A$.

Take one of the angles A. Then of the other two, one must be acute. Let B be an acute angle. From C draw CF perpendicular to BA, or to BA produced if necessary.

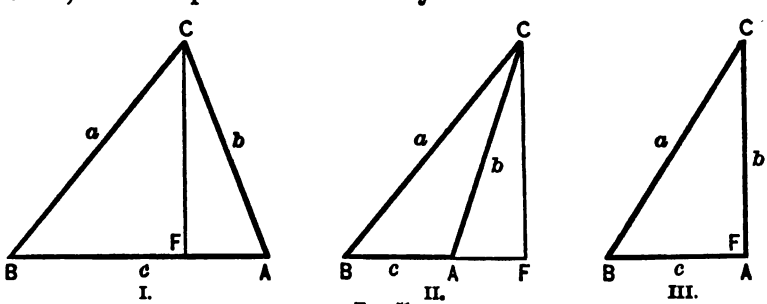


Fig. 51.

There will be three figures according as A is less, greater than, or equal to a right angle. Then,

I. Fig. 51. $BC^2 = CA^2 + AB^2 - 2 \cdot BA \cdot FA$; (Geom.)

or, $a^2 = b^2 + c^2 - 2\,c \cdot FA$

$= b^2 + c^2 - 2\,cb \cos A.$ (For $FA = b \cdot \cos A$.)

II. Fig. 51. $BC^2 = CA^2 + AB^2 + 2 \cdot BA \cdot AF$; (Geom.)

or, $a^2 = b^2 + c^2 + 2\,cb \cos FAC$

$= b^2 + c^2 - 2\,bc \cos A.$ (For $FAC = 180° - A$.)

III. Fig. 51. $BC^2 = CA^2 + AB^2$; (Geom.)

or, $a^2 = b^2 + c^2 - 2\,bc \cos A.$ (For $\cos A = \cos 90° = 0$.)

Similarly it may be proved that

$$b^2 = c^2 + a^2 - 2\,ca \cos B,$$

and that $$c^2 = a^2 + b^2 - 2\,ab \cos C.$$

108. V. Hence,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2\,ca}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

***109.** The formulæ of Art. 108 may be obtained directly from those of Art. 105.

$$a = b \cos C + c \cos B. \tag{1}$$

$$b = c \cos A + a \cos C. \tag{2}$$

$$c = a \cos B + b \cos A. \tag{3}$$

Multiplying (1), (2), (3) by a, b, c respectively and adding, we obtain

$$a^2 + b^2 + c^2 = 2\,a\,(b \cos C + c \cos B) + 2\,bc \cos A = 2\,a^2 + 2\,bc \cos A.$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

Exercise I. Find the two corresponding expressions, viz., for $\cos B$ and $\cos C$.

Exercise II. If $a = 5$, $b = 6$, $c = 7$, find $\cos A$.

110. VI. Let s stand for half the sum of a, b, c; so that

$$(a + b + c) = 2\,s.$$

Then, $(b + c - a) = (b + c + a - 2\,a) = (2\,s - 2\,a) = 2\,(s - a)$,

and $(c + a - b) = (c + a + b - 2\,b) = (2\,s - 2\,b) = 2\,(s - b)$,

and $(a + b - c) = (a + b + c - 2\,c) = (2\,s - 2\,c) = 2\,(s - c)$.

111. VII. To prove that

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}} \text{ and that } \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

where s stands for half the sum of the sides a, b, c.

Now, since

$$\cos A=\frac{b^2+c^2-a^2}{2\,bc} \text{ and } 1-\cos A=2\sin^2\frac{A}{2}, \quad \text{(Art. 88.)}$$

$$\therefore\ 2\sin^2\frac{A}{2}=1-\cos A=1-\frac{b^2+c^2-a^2}{2\,bc}$$

$$=\frac{2\,bc-(b^2+c^2-a^2)}{2\,bc}=\frac{a^2-(b^2-2\,bc+c^2)}{2\,bc}$$

$$=\frac{a^2-(b-c)^2}{2\,bc}=\frac{\{a-(b-c)\}\{a+(b-c)\}}{2\,bc};$$

$$\therefore\ \sin^2\frac{A}{2}=\frac{(a+c-b)(a+b-c)}{4\,bc};$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(2\,s-2\,b)(2\,s-2\,c)}{4\,bc}}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Again, since $2\cos^2\frac{A}{2}=1+\cos A$, (Art. 88.)

$$\therefore\ 2\cos^2\frac{A}{2}=1+\cos A=1+\frac{b^2+c^2-a^2}{2\,bc}$$

$$=\frac{(b+c)^2-a^2}{2\,bc}=\frac{(b+c+a)(b+c-a)}{2\,bc};$$

$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{2\,s\cdot(2\,s-2\,a)}{4\,bc}}=\sqrt{\frac{s(s-a)}{bc}}.$$

112. $$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

EXAMPLE. Write down the corresponding formulæ for

$$\sin\frac{B}{2}, \text{ for } \cos\frac{B}{2}, \text{ and for } \tan\frac{B}{2}.$$

113. VIII. Again,

$$\sin A=2\sin\frac{A}{2}\cdot\cos\frac{A}{2}; \quad \text{(Art. 91.)}$$

$$\therefore\ \sin A=2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore\ \sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

The letter S usually stands for $\sqrt{s(s-a)(s-b)(s-c)}$, so that the above may be written $\frac{\sin A}{a}=\frac{2S}{abc}$.

Similarly, $$\frac{\sin B}{b}=\frac{2S}{abc}=\frac{\sin C}{c}.$$

114. IX. To prove that

$$\frac{b-c}{b+c}\cdot\cot\frac{A}{2}=\tan\frac{B-C}{2}.$$

Since $\frac{b}{\sin B}=\frac{c}{\sin C}$ (Art. 106), let each of these fractions $=d$.

Then $b=d\sin B$, and $c=d\sin C$.

$$\therefore\ \frac{b-c}{b+c}=\frac{d\sin B-d\sin C}{d\sin B+d\sin C}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

$$=\frac{2\sin\frac{B-C}{2}\cdot\cos\frac{B+C}{2}}{2\sin\frac{B+C}{2}\cdot\cos\frac{B-C}{2}}$$

$$=\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} \qquad \text{(Art. 82.)}$$

$$\therefore\ \frac{b-c}{b+c}=\frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}. \qquad \left[\text{Since } \tan\frac{B+C}{2}=\tan\left(90^\circ-\frac{A}{2}\right).\right]$$

$$\therefore\ \frac{b-c}{b+c}\cdot\cot\frac{A}{2}=\tan\frac{B-C}{2}. \qquad \text{Q.E.D.}$$

Similarly, $$\frac{c-a}{c+a}\cot\frac{B}{2}=\tan\frac{C-A}{2};$$

$$\frac{a-b}{a+b}\cot\frac{C}{2}=\tan\frac{A-B}{2}.$$

115. The student is advised to make himself thoroughly familiar with the following formulæ:

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2S}{bc}; \qquad \text{(i.) (Art. 113.)}$$

$$a=b\cos C+c\cos B; \qquad \text{(ii.) (Art. 105.)}$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{abc}{2S}; \qquad \text{(iii.) (Art. 106.)}$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}; \qquad \text{(iv.) (Art. 108.)}$$

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}; \qquad \text{(v.) (Art. 111.)}$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}; \qquad \text{(vi.) (Art. 111.)}$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}; \qquad \text{(vii.) (Art. 112.)}$$

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}. \qquad \text{(viii.) (Art. 114.)}$$

116. The sign of the radicals in v., vi., and vii. of Art. 115 is positive, because $\frac{A}{2}<90°$.

$\operatorname{Sin}\frac{A}{2}$, $\cos\frac{A}{2}$, $\tan\frac{A}{2}$ cannot be imaginary, since, then, either $s-a$, $s-b$, $s-c$, is negative; which is impossible.

EXAMPLES. XXVII.

In any triangle ABC prove the following statements:

1. $\frac{\sin A+2\sin B}{a+2b}=\frac{\sin C}{c}$.
2. $\frac{\sin^2 A-m\cdot\sin^2 B}{a^2-m\cdot b^2}=\frac{\sin^2 C}{c^2}$.
3. $a\cos A+b\cos B-c\cos C=2c\cos A\cdot\cos B$.
4. $(a+b)\sin\frac{C}{2}=c\cos\frac{A-B}{2}$.
5. $(b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}$.
6. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$.
7. $\frac{a-b}{c}=\frac{\cos B-\cos A}{1+\cos C}$.
8. $\frac{b+c}{a}=\frac{\cos B+\cos C}{1-\cos A}$.
9. $\sqrt{bc\sin B\cdot\sin C}=\frac{b^2\sin C+c^2\sin B}{b+c}$.
10. $a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C$.
11. $b+c-a=(b+c)\cos A-(c-a)\cos B+(a-b)\cos C$.
12. $\tan A=\frac{a\sin C}{b-a\cos C}$.
13. $\frac{\tan B}{\tan C}=\frac{a^2+b^2-c^2}{a^2-b^2+c^2}$.

In solving the following list of examples, the student will select from the formulæ of Art. 115 those best suited to his purpose.

EXAMPLE. Given $a=2$, $b=\sqrt{6}$, $c=1+\sqrt{3}$. To find the angles of the triangle.

It is evident that we may apply either iv., v., vi., or vii.; but that ii., iii., and viii. contain *two* angles, and hence cannot be used. We shall employ iv. We find

$$\cos A=\frac{1}{\sqrt{2}},\quad \therefore\ A=45°;$$

$$\cos B=\tfrac{1}{2},\quad \therefore\ B=60°;$$

$$C=180°-(A+B)=75°.$$

EXAMPLES. XXVIII.

1. Simplify the formulæ

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}, \quad \cos \tfrac{1}{2} A = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$$

in the case of an equilateral triangle.

2. The sides of a triangle are as $2 : \sqrt{6} : 1 + \sqrt{3}$; find the angles.

3. The sides of a triangle are as $4,\ 2\sqrt{2},\ 2(\sqrt{3} - 1)$; find the angles.

4. Given $C = 120°$, $c = \sqrt{19}$, $a = 2$; find b.

5. Given $A = 60°$, $b = 4\sqrt{7}$, $c = 6\sqrt{7}$; find a.

6. Given $A = 45°$, $B = 60°$, and $a = 2$; find c.

7. The sides of a triangle are as $7 : 8 : 13$; find the greatest angle.

8. The sides of a triangle are $1,\ 2,\ \sqrt{7}$; find the greatest angle.

9. The sides of a triangle are as $a : b : \sqrt{(a^2 + ab + b^2)}$; find the greatest angle.

10. When $a : b : c$ as $3 : 4 : 5$, find the greatest and least angles; given $\cos 36° 52' = .8$.

11. If $a = 5$ miles, $b = 6$ miles, $c = 10$ miles, find the greatest angle. [$\cos 49° 33' = .65$.]

12. If $a = 4$, $b = 5$, $c = 8$, find C. Given that $\cos 54° 54' = .575$.

13. $a : b = \sqrt{3} : 1$, and $C = 30°$; find the other angles.

14. If $b = 3$, $C = 120°$, $c = \sqrt{13}$, find a and the sines of the other angles.

15. Given $A = 105°$, $B = 45°$, $c = \sqrt{2}$; solve the triangle.

16. Given $B = 75°$, $C = 30°$, $c = \sqrt{8}$; solve the triangle.

17. Given $B = 45°$, $c = \sqrt{75}$, $b = \sqrt{50}$; solve the triangle.

18. Two sides of a triangle are $3\sqrt{6}$ yards and $3\sqrt{3} + 1$ yards, and the included angle is $45°$; solve the triangle.

19. If the angles adjacent to the base of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$, show that the perpendicular altitude will equal half the base.

20. If $A = 45°$ and $B = 60°$, show that $2\,c = a(1 + \sqrt{3})$.

21. The cosines of two of the angles of a triangle are $\frac{1}{2}$ and $\frac{2}{3}$; find the ratio of the sides.

CHAPTER XIV

LOGARITHMS

117. Before proceeding to the problem known as the solution of triangles, we shall discuss *very briefly* the use of common logarithms and certain mathematical tables. These tables may be found on the last sixty-five pages of this book.

NOTE. — The discussion of logarithms belongs properly to Algebra, to which the student is referred for a more general treatment.

The student will observe that computation by means of logarithms is a mere combination of exponents.

118. In Algebra it is explained that when different powers or roots of the same number are concerned,

(i.) *multiplication* is effected by *adding* the *indices;*

(ii.) *division* is effected by *subtracting* the *indices;*

(iii.) *involution* and *evolution* are respectively effected by the *multiplication* and *division* of the *indices.*

EXAMPLE 1. Let $m = a^h$, $n = a^k$;

then
$$m \times n = a^h \times a^k = a^{h+k}, \qquad \text{(i.)}$$
$$m \div n = a^h \div a^k = a^{h-k}, \qquad \text{(ii.)}$$
$$\left.\begin{aligned} m^3 &= (a^h)^3 = a^{3h}, \\ \sqrt[4]{m} &= m^{\frac{1}{4}} = (a^h)^{\frac{1}{4}} = a^{\frac{h}{4}}. \end{aligned}\right\} \qquad \text{(iii.)}$$

119. DEFINITION. The **logarithm of a number,** n, to a certain **base,** b, is the index with which it is necessary to affect b to produce n.

E.g. suppose $b^l = n$; then logarithm n to base b is l for b, raised to lth power, produces n. This is expressed in the following **notation:**

$$\log_b n = l.$$

and is read "logarithm of n to the base b equals l." Or, if no ambiguity arises, simply "$\log n = l$."

EXAMPLES. $\log_2 8 = 3$; for $2^3 = 8$, or 2 must be raised to power 3 to produce 8.

$$\log_{10} 100 = 2.$$

120. The use of logarithms is based upon the following propositions:

I. *The logarithm of the* **product** *of two numbers is equal to the logarithm of one of the numbers plus the logarithm of the other.*

For, let $\log_b m = x$; then $m = b^x$, (Def.)

and let $\log_b n = y$; then $n = b^y$. (Def.)

$$(mn) = (b^x b^y) = (b^{x+y}).$$

$$\therefore \log_b(mn) = \log_b(b^{x+y}) = x + y.$$

But $x + y = \log_b m + \log_b n$;

or, $\log_b(mn) = x + y = \log_b m + \log_b n.$ Q.E.D.

II. *The logarithm of the* **quotient** *of two numbers is the logarithm of the* **dividend** *minus the logarithm of the* **divisor.**

For $\left(\frac{m}{n}\right) = \left(\frac{b^x}{b^y}\right) = b^{(x-y)}.$

But $x - y = \log_b m - \log_b n.$

$$\therefore \log_b\left(\frac{m}{n}\right) = x - y = \log_b m - \log_b n.$$

III. The logarithm of a number raised to a power k is k times the logarithm of the number.

For, $(m^k) = (b^x)^k.$

Now, $\log_b b^{kx} = kx = k \log_b m.$

Examples. Given $\log_{10} 2 = .30103$, $\log_{10} 3 = .47712$, $\log_{10} 7 = .84509$; find the values of the following:

(i.) $\log_{10} 6 = \log_{10}(2 \times 3) = \log_{10} 2 + \log_{10} 3$

$= .30103 + .47712 = .77815.$ [by I.]

(ii.) $\log_{10} \frac{7}{3} = \log_{10} 7 - \log_{10} 3 = .84509 - .47712 = .36797.$ [by II.]

(iii.) $\log_{10} 3^5 = 5$ times $\log_{10} 3 = 5 \times .30103 = 1.50515.$ [by III.]

(iv.) $\log_{10}\sqrt[3]{\frac{3 \times 4}{7}} = \log_{10}\left(\frac{3 \times 4}{7}\right)^{\frac{1}{3}} = \frac{1}{3}$ of $\log_{10}\frac{3 \times 4}{7}$ [by III.]

$= \frac{1}{3}$ of $(\log 3 + \log 4 - \log 7) = \frac{1}{3}$ of $\{.47712 + \text{twice } .30103 - .84509\}$

$= \frac{1}{3}$ of $.23408 = .07802.$ [by I. and II.]

(v.) $\log_{10} 5 = \log_{10} \frac{10}{2} = \log_{10} 10 - \log_{10} 2 = 1 - .30103 = .69897.$

EXAMPLES. XXIX.

1. Find the logarithms to the base a of a^3, $a^{\frac{10}{3}}$, $\sqrt[4]{a}$, $\sqrt[3]{a^2}$, $\frac{1}{a^{\frac{5}{2}}}$.
2. Find the logarithms to the base 2 of 8, 64, $\frac{1}{2}$, .125, .015625, $\sqrt[3]{64}$.
3. Find the logarithms to the base 3 of 9, 81, $\frac{1}{3}$, $\frac{1}{27}$, $\frac{1}{81}$.

4. Find the value of $\log_2 8$, $\log_2 .5$, $\log_3 243$, $\log_5 .04$, $\log_{10} 1000$, $\log_{10} .001$.

5. Find the value of $\log_a a^{\frac{4}{3}}$ $\log_b \sqrt[3]{b^2}$ $\log_3 2$, $\log_{27} 3$, $\log_{100} 10$.

6. Prove that $\log(\sqrt[3]{2} \times \sqrt[4]{7} \div \sqrt[5]{9}) = \frac{1}{3}\log 2 + \frac{1}{4}\log 7 - \frac{2}{5}\log 3$.

121. That system of logarithms whose base is 10 is called the **common** system of logarithms.

In speaking of logarithms hereafter, *common* logarithms are referred to unless the contrary is expressly stated.

We shall assume that an index of 10 can be found such that 10 affected with this index is practically equivalent to any number.

The indices of these powers of 10, *i.e.* the common logarithms, are in general *incommensurable* numbers.

Now, the greater the index with which 10 is affected, the greater will be the value of the equivalent expression; and the less the index, the less will be the numerical value of the expression.

Hence, if one number be less than another, the logarithm of the first will be less than the logarithm of the second.

But the student should notice that logarithms (or indices) are *not proportional* to the corresponding numbers.

EXAMPLE. 1000 is less than 10000; and the logarithm to base 10 of the first is 3 and of the second is 4.

But 1000, 10000, 3, 4 are not in proportion.

122. PROPOSITION. *If two numbers expressed in the decimal notation have the same digits arranged in the same order (so that they differ only in the position of the decimal point), their logarithms to the base* 10 *differ only by an integer.*

The decimal point in a number is moved by multiplying or dividing the number by some power of 10.

Let the numbers be m and n; then $m = n \times 10^k$ when k is a whole number (positive or negative); then

$$\log m = \log(n \times 10^k) = \log n + \log 10^k = \log n + k. \quad \text{(Art. 120.)}$$

That is, $\log m$ and $\log n$ differ by an integer. Q.E.D.

EXAMPLE 1. $\log 1679.2 = \log\{(1.6792) \times 10^3\} = \log 1.6792 + \log 10^3$
$= \log 1.6792 + 3.$

EXAMPLE 2. Given that $\log 1.7692 = .247776$;

find (i.) $\log 17692$, (ii.) $\log .0017692$, (iii.) $\log 176.92$.

Here $\log 17692 = \log(1.7692 \times 10^4) = 4.247776,$

$\log .0017692 = \log(1.7692 \times 10^{-3}) = -3 + .247776,$

$\log 176.92 = \log(1.7692 \times 10^2) = 2.247776.$

123. We know from Algebra that $1 = 10^0$,

$$10 = 10^1 \text{ and that } .1 = \tfrac{1}{10} = 10^{-1}$$
$$100 = 10^2 \text{ and that } .01 = \tfrac{1}{100} = 10^{-2}$$
$$1000 = 10^3 \text{ and that } .001 = \tfrac{1}{1000} = 10^{-3}$$
$$10000 = 10^4 \text{ and that } .0001 = \tfrac{1}{10000} = 10^{-4}$$

and so on.

Hence, the logarithm of 1 is 0.

The logarithm of any number greater than 1 is *positive.*

The logarithm of any **positive** number less than 1 is *negative.*

124. Observe also

that the logarithm of any number between 1 and 10 is a positive decimal fraction;

that the logarithm of any number between 10 and 100, *i.e.* between 10^1 and 10^2, is $1 +$ a decimal fraction;

that the logarithm of any number between 1000 and 10000, *i.e.* between 10^3 and 10^4, is $3 +$ a decimal fraction; and so on.

125. Observe also

that the logarithm of any number between 1 and .1, *i.e.* between 10^0 and 10^{-1}, can be written in the form $-1 +$ **a decimal fraction;**

that the logarithm of any number between .1 and .01, *i.e.* between 10^{-1} and 10^{-2}, can be written in the form $-2 +$ **a decimal fraction;** and so on.

EXAMPLE 1. How many digits are contained in the integral part of the number whose logarithm is 3.67192 ?

The number is $10^{3.67192}$ and this is greater than 10^3, *i.e.* greater than 1000, and it is less than 10^4, *i.e.* less than 10000. Therefore the number lies between 1000 and 10000, and therefore the integral part of it contains four figures.

EXAMPLE 2. Given that $3 = 10^{.4771213}$, find the number of the digits in the integral part of 3^{20}.

We have $$3 = 10^{.4771213},$$
$$\therefore\ 3^{20} = (10^{.4771213})^{20} = 10^{9.54243}.$$

Therefore there are 10 digits in the integral part of 3^{20}; for it is greater than 10^9 and less than 10^{10}.

EXAMPLE 3. Suppose that the decimal part of the logarithm is to be kept positive, find the integral part of the logarithm of .0001234.

This number is greater than .0001, *i.e.* than 10^{-4} and less than .001, *i.e.* than 10^{-3}.

Therefore its logarithm lies between -3 and -4, and therefore it is $-4 +$ a fraction; the integral part is therefore -4.

126. From Art. 120–125 it is evident that the logarithm of any positive number may be written as an **integer** + a decimal fraction.

The *integral* part of the logarithm is called the **characteristic.** The *decimal* part of the logarithm is called the **mantissa.** For *convenience,* the mantissæ of common logarithms are always kept **positive.** In this way the mantissæ of the logarithms of numbers consisting of the same digits, arranged in the same order, are *always* the *same* (Art. 120); because removing the decimal place to the right or to the left is equivalent to multiplying the number by 10^k, where k is a positive or negative *integer,* as the case may be.

EXAMPLE. The mantissa of $\log 3.456$ = mantissa of $\log (345.6)$.

The student cannot observe too carefully that the mantissa is always positive.

The mantissæ have been calculated and arranged in convenient tables. See table I.

127. It is evident from Arts. 120–125, that the characteristic of a logarithm can be obtained by the following rule:

RULE. The characteristic of the logarithm of a number greater than unity is **one less** than the number of figures in the integral part of the number.

The characteristic of a number less than unity is negative, and (when the number is expressed as a decimal) is **one more** than the number of ciphers between the decimal point and the first significant figure to the right of the decimal point.

When the characteristic is negative, as for example in the logarithm $-3 + .17609$, the logarithm is abbreviated thus, $\bar{3}.17609$.

EXAMPLE 1. The characteristics of 36741, 36.741, .0036741, 3.6741, and .36741 are respectively 4, 1, -3, 0, and -1.

EXAMPLE 2. Given that the mantissa of the logarithm of 36741 is 56515, we can at once write down the logarithm of any number whose digits are 36741.

Thus

$$\log 3674100 = 6.56515,$$
$$\log 36741 = 4.56515,$$
$$\log 367.41 = 2.56515,$$
$$\log .36741 = \bar{1}.56515,$$
$$\log .00036741 = \bar{4}.56515,$$

and so on.

128. We have said that logarithms are in general incommensurable numbers. Their values can, therefore, only be given approximately.

If the value of any number is given to seven significant figures, then the error (*i.e.* the difference between the *given* value and the *exact* value of the number) is less than a millionth part of the number.

EXAMPLE. 3.141592 is the value of π correct to seven significant figures. The *error* is less than .000001; for π is less than 3.141593, and greater than 3.141592.

The ratio of .000001 to 3.141592 is equal to 1 : 3141592. The ratio of .000001 to π is less than this; *i.e.* much less than the ratio of one to one million.

129. An actual measurement of any kind must be made with the greatest care, with the most accurate instruments, by the most skilful observers, if it is to attain to anything like the accuracy represented by 'seven significant figures'; and, indeed, the value of any quantity given correct to 'four significant figures' is exact for most practical purposes.

130. A five-place table of logarithms is placed at the end of the book. (See table I.) Page 1 of this table contains the logarithms, to five places of decimals, of all numbers from 1 to 100. Pages 2–16 contain the mantissæ, to five decimal places, of the logarithms all numbers from 100 to 10000. But all numbers from 0 to ∞ is one of these numbers multiplied by ten affected with either a positive or negative index,

$$e.g.\ 4628326 = 4.628326 \times 10^6;\ \ .03986 = 3.986 \times 10^{-2}.$$

Hence by prefixing to the mantissæ the proper characteristic (see Art. 126) we obtain the logarithm of any number, of not more than four significant figures, from 0 to ∞.

131. *To find the logarithm of a given number.* (*a*) If the number contains not more than four significant figures. Find the mantissæ from the table corresponding to these four significant figures and prefix the proper characteristic. The result is the logarithm required.

EXAMPLE 1. To find the logarithm of 4064.

$$4064 = 4.064 \cdot 10^3.$$

Referring to the table I., page 8, we find, at the intersection of the row headed 406 and the column headed 4, the number .60895.

$\therefore$ log 4064. = 3.60895.

EXAMPLE 2. To find log .04064. This logarithm differs from the former as to the characteristic, which is -2.

$\therefore$ log .04064 = $\bar{2}$.60895.

(*b*) To find the logarithm of a number of more than four digits.

We shall **assume** that *if the difference is small the difference between numbers is proportional to the difference of their logarithms.* This proposition is proved in works on Algebra. If there are more than four digits in the number, we cannot obtain its logarithm directly from the table, but must *interpolate.* This is illustrated by the following example. To find the logarithm of 3456.4. This number lies between 3457 and 3456. Its logarithm therefore lies between log 3457 and log 3456.

$$\log 3457 - \log 3456 = 3.53870 - 353857 = .00013.$$

If $\log 3456.4 = l$,

$$\log 3456.4 - \log 3456 = l - 3.53857.$$

By the theorem stated above,

$$1 : .00013 = .4 : l - 3.53857;$$
$$\therefore\ l = .4 \times .00013 + 3.53857;$$
$$\therefore\ \log 3456.4 = 3.53862.$$

Or we may reason thus:

Since increasing 3456 by 1 increases its logarithm by .00013 increasing 3456 by .4 of 1 increases its logarithm by .4 of .00013 or by .000052.

In forming such products as $.4 \times .00013$, we retain only five decimal places. We increase the number occupying the fifth place by unity if the succeeding number is equal to or is greater than 5. We neglect the number occupying the sixth place if it is less than 5.

Example. To find the logarithm of 56.452.

$$\begin{array}{r} \log 56.46 = 1.75174 \\ \log 56.45 = \underline{1.75166} \\ 8 \end{array}$$

$$.2 \times 8 = 1.6. \quad \therefore\ \log 56.452 = 1.75168.$$

EXAMPLES. XXX.

Find logarithms of the following numbers:

1. 3562; 7.456; .00432.
2. 86421; 96.204; .00352.
3. .045624; .035421; .0072345.
4. 6789000; 32,456,000.

132. *To find the number whose logarithm is given.* The method of procedure is just the reverse of that of Art. 131, and will be illustrated by the following examples:

Example 1. To find the number whose logarithm is 3.41447.

Looking in the body of the table of logarithms of numbers, we find, at the intersection of the line headed 259 and the column headed 7, the given mantissa 41447.

$\therefore$ log 2.597 = .41447. The characteristic 3 shows there are four figures to the left of the decimal point. Hence the number required is 2597.

Example 2. The number whose logarithm is 5.41447 is 259700.

The number whose logarithm is $\bar{2}.41447$ is .02597.

It may happen that the mantissa of the given logarithm does not occur in the table. We then proceed as follows:

To find the number whose logarithm is .43563.

The mantissæ next lower and next higher than the given mantissa are .43553 and .43569 respectively.

$$\log 2.727 = .43569.$$
$$\log 2.726 = .43553.$$
$$.43569 - .43553 = .00016.$$
$$.43562 - .43553 = .00009.$$

Using the proportion of Art. 132,

$$.001 : .00016 = n - 2.726 : .00009,$$

where n is the desired number;

$$\therefore\ n = 2.72656.$$

Or thus:

Since the mantissa of the number is .00009 greater than the mantissa of log 2.726, and the mantissa of log 2.727 is .00016 greater, the number must be $\frac{9}{16}$ of .001 greater than 2.726.

The number obtained is not accurate to more than five significant figures. If the characteristic indicates that more than five figures are to the left of the decimal point, the remaining places are filled with ciphers. Thus in the problem solved we can rely only on 27265. The 6 is questionable.

EXAMPLES. XXXI.

1. Find the number whose logarithm is .56867.
2. Find the number whose logarithm is 4.66029.
3. Find the number whose logarithm is 6.39669.
4. Find the number whose logarithm is $\bar{4}.64311$.
5. Find the number whose logarithm is .75504.

EXAMPLES. XXXII.

Find the values of the following correct to four significant figures:

1. $\sqrt[3]{451}$.
2. $\sqrt[5]{802}$.
3. $(273)^{\frac{4}{9}} \times (234)^{\frac{1}{4}}$.
4. $(451)^{\frac{3}{5}} \times (231)^{\frac{4}{7}}$.
5. $\left(\frac{192.5}{84}\right)^{3}$.
6. $\frac{(34.79)^{\frac{2}{3}}}{(41.25)^{\frac{3}{2}}}$.
7. $\frac{(24.76)^{\frac{2}{7}}}{(.0045)^{\frac{3}{2}}}$.
8. $\frac{7.89}{.0345} \times (89130)^{\frac{1}{7}}$.
9. $\frac{\frac{3}{2}\sqrt{(5.2)}}{5\sqrt{(11.31)}} \times (\frac{3}{7})^{-\frac{1}{2}}$.
10. $\sqrt[5]{\left\{\frac{2\sqrt{(34)}}{3\sqrt{(791)}}\right\}}$.
11. $\frac{\sqrt[4]{3}}{\sqrt[6]{3}}$
12. $\left(\frac{21^{3} \times 45^{5}}{2^{7} \times 3^{9}}\right)^{\frac{1}{2}}$.

Solve the equations correct to four figures:

13. $10^{x} = 421$.
14. $(\frac{31}{40})^{x} = 3$.
15. $(\frac{303}{200})^{2x} = 2$.

133. Since the trigonometric ratios are numbers we can find their logarithms. Since the sine of an angle is never greater than 1, the characteristic of its logarithm is negative (except for $\sin\theta = 1$).

To avoid the use of negative characteristics it is usual to add ten to the actual logarithm of $\sin\theta$ and call the result $\log\sin\theta$, *e.g.* the actual logarithm of $\sin 7°$ is $\bar{1}.08589$, but as explained above it is written $\log\sin 7° = 9.08589$. Similar remarks apply to the logarithm of the cosine of an angle, to the logarithm of tangents of angles from 0° to 45°, and to the logarithm of cotangents of angles from 45° to 90°.

Table II. contains the logarithmic sine, tangent, cotangent, and cosine for every ten seconds from 0° to 2°, and for every minute from 1° to 89°. From these we can find the logarithms of the trigonometric ratios of any angle, because (Art. 58) the ratios of any angle can be expressed in terms of the ratios of angles of the first quadrant.

134. *To find $\log\sin\theta$, having given θ.* We shall illustrate the method of procedure by some examples.

EXAMPLE 1. To find $\log\sin 15° 25'$, see tables, page 41, at the intersection of the line headed 25, in left margin, and the column headed $\log\sin\theta$ under 15°, we find 9.42416, *i.e.* $\log\sin 15° 25' = 9.42461$.

EXAMPLE 2. To find $\log\sin 74° 20' 40''$.

Log $\sin 74° 20' 40''$ cannot be found directly in the tables. Hence we must *interpolate*.

We assume the theorem, *A very small change in an angle is proportional to the corresponding change in its sine.*[1]

[1] We have added a proof of this theorem in Arts. 137–139.

From the tables, at the intersection of the line headed 20 in the *right* margin and column headed log sin θ *above* 74°, we find 9.98356, *i.e.*

$$\begin{aligned} \log \sin 74^\circ 20' &= 9.98356 \\ \text{Similarly } \log \sin 74^\circ 21' &= \underline{9.98359} \\ &\ \ 3 \end{aligned}$$

Hence the difference of 1′ (*i.e.* 60″) in the angle corresponds to a difference .00003 in the log sin of its angle. Hence by theorem quoted above, to a difference of 40″ in the angle corresponds a difference of $\frac{40}{60}$ of .00003 = .00002.

$$\begin{aligned} \therefore \log \sin 74^\circ 20' 40'' &= 9.98356 + .00002 \\ &= 9.98358. \end{aligned}$$

135. To find θ, having given log sin θ. This is the converse of Art. 134. The reasoning is not essentially different from that of Art. 131.

136. The sine, cosine, etc., are sometimes called the natural sine, natural cosine, etc. Table III., *a*, *b*, *c*, *d*, contains the natural trigonometric functions from 0° to 90° at intervals of 6′.

***137.** To prove $\sin\theta < \theta \tan\theta$. Let $P'XP$ be a circle with centre O, and radius OP. Draw the tangents PI and $P'I$. Connect PP'.

Let θ be the *circular* measure of XOP. Then 2θ is the circular measure of $P'OP$. Now

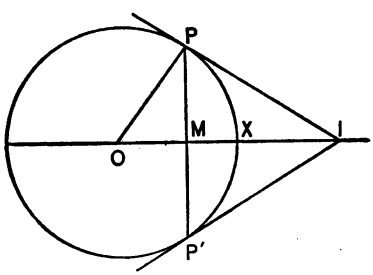


FIG. 52.

$$PP' < \text{arc } PP < (PI + IP'). \qquad \text{(Geom.)}$$

$$\therefore PM < \text{arc } XP < PI.$$

$$\frac{PM}{OP} < \frac{\text{arc } XP}{OP} < \frac{PI}{OP}. \qquad (1)$$

But $\qquad \text{arc } XP = OP \cdot \theta.$

Substituting in (1), $\quad \sin\theta < \theta < \tan\theta. \qquad (2)$ Q.E.D.

***138.** *To prove when θ is very small.*

$$\frac{\sin\theta}{\theta} = 1, \text{ approximately;}$$

$$\frac{\tan\theta}{\theta} = 1, \text{ approximately.}$$

Divide (2) of the last article by $\sin\theta$.

$$1 < \frac{\theta}{\sin\theta} < \cos\theta.$$

By Art. 39, when θ is very small, $\cos\theta$ approaches 1.

But $\dfrac{\theta}{\sin\theta}$ lies between 1 and $\cos\theta$.

Hence, approximately, $\dfrac{\theta}{\sin\theta} = 1$ when θ is *very small.*

$$\frac{\theta}{\tan\theta} = \frac{\theta}{\sin\theta}\cos\theta. \quad \text{Hence, as before,}$$

$$\frac{\theta}{\tan\theta} = 1 \text{ when } \theta \text{ is } \textit{very small.}$$

It must not be forgotten that θ is given in circular measure.

***139.** $$\frac{\sin(\theta + x) - \sin\theta}{\sin(\theta + y) - \sin\theta} = \frac{\sin\theta\cos x + \cos\theta\sin x - \sin\theta}{\sin\theta\cos y + \cos\theta\sin y - \sin\theta}.$$

If now x is very small, and y is very small, we may write $\cos x = 1$, $\sin x = x$; and if y is very small, $\cos y = 1$, $\sin y = y$.

$$\therefore \text{ when } x \text{ and } y \text{ are very small, } \frac{\sin(\theta + x) - \sin\theta}{\sin(\theta + y) - \sin\theta} = \frac{x}{y}. \qquad (1)$$

Or, a very small change in an angle is proportional to the corresponding change in the sine of that angle.

In a similar way this law of proportional change may be established for each of the natural functions.

CHAPTER XV

ON THE SOLUTION OF TRIANGLES

140. The problem known as the **solution of triangles** may be stated thus: *When a sufficient number of the parts of a triangle are given, to find the magnitude of each of the other parts.*

141. Solution of the right triangle (see Ch. VI.).

Let ACB be a right-angled triangle with the right angle at C.

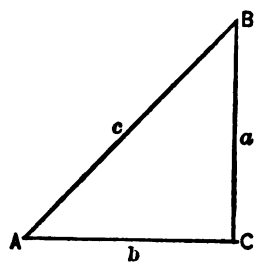


FIG. 58.

We have

$$a = c \sin A = c \cos B;$$

$$b = c \cos A = c \sin B;$$

$$a = b \tan A = b \cot B.$$

The signs of the ratios are all positive.

$$\therefore \log a = \log c + \log \sin A - 10; \qquad \text{(Art. 132.)}$$

$$\log b = \log c + \log \cos A - 10; \qquad \text{(Art. 132.)}$$

$$\log a = \log b + \log \tan A.^{1} \qquad \text{(Art. 132.)}$$

EXAMPLES. XXXIII.

1. Given $a = 12562$, $A = 12°$. Find B, b, and c.
2. Given $c = 35$, $B = 37° 10' 5''$. Solve the triangle.
3. Given $b = 100$, $B = 42°$. Find c and a.

[1] See footnote, page 94.

142. The student has proved, while studying Geometry, that a triangle is uniquely determined when there are given:

I. Three sides.

II. One side and two angles.

III. Two sides and the included angle. And

IV. That either one, two, or no triangles are determined when two sides and the angle opposite one of them is given.

When, therefore, three parts (one of which is a side) are given, the other parts can be calculated. There are four cases.

Case I.

143. Given **three sides**, a, b, c. (Art. 141, I.)

We find two of the angles from the formulæ

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}.$$

The third angle $C=180^\circ-A-B$.

144. In practical work we proceed as follows:

$$\log\tan\frac{A}{2}=\log\sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

or,

$$\log\tan\frac{A}{2}=\tfrac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}.$$

Similarly,

$$\log\tan\frac{B}{2}=\tfrac{1}{2}\{\log(s-c)+\log(s-a)-\log s-\log(s-b)\}.$$

145. Either of the formulæ

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

may also be used as above.

The $\sin\frac{A}{2}$ and the $\cos\frac{A}{2}$ formulæ are either of them as convenient as the $\tan\frac{A}{2}$ formula, when *one* of the angles only is to be found. If *all* the angles are to be found, the tangent formulæ are convenient, because we can find the log tangents of two half angles from the same *four* logs, viz. $\log s$, $\log(s-a)$, $\log(s-b)$, $\log(s-c)$. To find the

log sines of *two* half angles we require the *six* logarithms, viz. $\log(s-a)$, $\log(s-b)$, $\log(s-c)$, $\log a$, $\log b$, $\log c$.

EXAMPLE. Given $a = 275.35$, $b = 189.28$, $c = 301.47$ chains; find A and B.

Here, $s = 383.05$, $s - a = 107.70$, $s - b = 193.77$, $s - c = 81.58$.

$$\text{Then } \log\tan\frac{A}{2} = \tfrac{1}{2}\{\log 193.77 + \log 81.58 - \log 383.05 - \log 107.70\}$$
$$= \tfrac{1}{2}\{2.28728 + 1.91158 - 2.58325 - 2.03221\}$$
$$= 9.79169^{1} \text{ (Art. 132)}, \qquad \text{[From the tables.]}$$

whence $\frac{A}{2} = 31^\circ\, 45'\, 28.5''$; $\therefore A = 63^\circ\, 30'\, 57''$. Again,

$$\log\tan\frac{B}{2} = \tfrac{1}{2}\{\log 81.58 + \log 107.70 - \log 383.05 - \log 193.77\}$$
$$= 9.53663 = \log\tan 18^\circ\, 59'\, 9.8'';$$
$$\therefore B = 37^\circ\, 58'\, 20''; \; C = 180^\circ - A - B = 78^\circ\, 30'\, 43''.$$

146. This case may also be solved by the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

But this formula is not adapted for logarithmic calculation, and therefore is seldom used in practice.

It may sometimes be used with advantage, when the given lengths of a, b, c are small.

EXAMPLE. Find the greatest angle of the triangle whose sides are 13, 14, 15.

Let $a = 15$, $b = 14$, $c = 13$. Then the greatest angle is A.

$$\text{Now,} \quad \cos A = \frac{14^2 + 13^2 - 15^2}{2 \times 14 \times 13} = \frac{140}{2 \times 14 \times 13} = \frac{5}{13} = .384615$$
$$= \cos 67^\circ\, 23', \text{ nearly.} \qquad \text{[By the table of natural cosines.]}$$
$$\therefore \text{ the greatest angle} = 67^\circ\, 23'.$$

EXAMPLES. XXXIV.

1. If $a = 352.25$, $b = 513.27$, $c = 482.68$ yards, find the angles A and B.

2. Find the two largest angles of the triangle whose sides are 484, 376, 522 feet.

3. Given $a = .641$, $b = .529$, $c = .702$; find two largest angles of the triangle.

4. Given $a = 2$, $b = 1.64$, $c = 2.075$; find B.

5. If $a = 3811$, $b = 3850$, $c = 4090$ yards, find C.

6. The sides are 2, $\sqrt{2}$, and $\sqrt{3} - 1$; find the angles.

[1] The result here is really $\bar{1}.79169$, but we have added ten to it in order to make this number agree with the one given in the table (see Art. 132). In the example, Art. 148, each of the values of $\log\sin A$ and $\log\sin B$ are ten too large; since one is negative and the other positive the result is not affected. Little difficulty will be experienced, if careful attention is given to Art. 132.

Case II.

147. Given **one side** and **two angles**, as a, B, C.

First, $A = 180° - B - C$; which determines A.

Next, $$\frac{b}{\sin B} = \frac{a}{\sin A}, \text{ or, } b = \frac{a \cdot \sin B}{\sin A};$$

and, $$\frac{c}{\sin C} = \frac{a}{\sin A}, \text{ or, } c = \frac{a \cdot \sin C}{\sin A}.$$

These determine b and c.

148. In practical work we proceed as follows:

Since $$b = \frac{a \cdot \sin B}{\sin A},$$

$$\therefore \log b = \log \frac{a \cdot \sin B}{\sin A},$$

$$\therefore \log b = \log a + \log \sin B - \log \sin A.$$

Similarly, $\log c = \log a + \log \sin C - \log \sin A$.

Example. Given that $c = 1764.3$ feet, $C = 18° 27'$, and $B = 66° 39'$; find b.

From the tables we find $\log 1764.3 = 3.24657$.

$$\log \sin 18° 27' = 9.50034, \log \sin 66° 39' = 9.96289;$$

$$\therefore \log b = 3.24657 + 9.96289 - 9.50034$$

$$= 3.70912 = \log 5118.2;$$

$$\therefore b = 5118.2 \text{ feet.}$$

EXAMPLES. XXXV.

1. $A = 53° 24'$, $B = 66° 27'$, $c = 338.65$ yards. Find a and C.
2. Find c, having given $a = 1000$, $A = 50°$, $C = 66°$.
3. Find b, having given $B = 32° 15'$, $C = 21° 47' 20''$, $a = 34$ feet.
4. Given $c = .0161$, $A = 35° 15'$, $C = 123° 39'$; solve the triangle.

Case III.

149. Given **two sides** and the **included angle**, as b, c, A.

First, $B + C = 180° - A$. Thus $(B + C)$ is determined.

Next, $$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}.$$

Thus $(B - C)$ is determined.

And B and C can be found when the values of $(B+C)$ and $(B-C)$ are known.

Lastly, $$\frac{a}{\sin A}=\frac{b}{\sin B}, \text{ or } a=\frac{b\cdot\sin A}{\sin B}.$$

Whence a is determined.

150. In practical work we proceed as follows:

Since $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2},$$

$$\therefore \log\left(\tan\frac{B-C}{2}\right)=\log(b-c)-\log(b+c)+\log\left(\cot\frac{A}{2}\right).$$

Also, since $$a=\frac{b\cdot\sin A}{\sin B},$$

$$\therefore \log a=\log b+\log\sin A-\log\sin B, \text{ as in Case II.}$$

EXAMPLE. Given $b=456.12$ chains, $c=296.86$ chains, and $A=74°\,20'$; find the other angles.

Here, $$b-c=159.26,\ b+c=752.98.$$

From the table we find

$$\log 159.26=2.2021, \text{ and } \log 752.98=2.87678,$$
$$\log\cot 37°\,10'=.1202;^{1}$$

$$\therefore \log\tan\frac{B-C}{2}=2.20210-2.87678+.12026=9.44558^{1}=\log\tan 15°\,35'\,18'';$$
$$\therefore B-C=31°\,10'\,36'', \text{ and } B+C=180°-74°\,20'.$$

Thus $$B+C=105°\,40';$$
$$\therefore 2B=136°\,50'\,36''; \ 2C=74°\,29'\,24'',$$
or $$B=68°\,25'\,18''; \text{ or, } C=37°\,14'\,42''.$$

151. The formula $a^2=b^2+c^2-2bc\cos A$ may be used in simple cases.

EXAMPLE. If $b=35$ feet, $c=21$ feet, and $A=50°$, find a, given that $\cos 50°=.643$.

Here $$a^2=35^2+21^2-2\times 35\times 21\times\cos 50°;$$

$$\therefore \frac{a^2}{7^2}=5^2+3^2-2\times 5\times 3\times\cos 50°,\ =25+9-30\times .643,\ =14.71.$$

$$\frac{a}{7}=3.82 \text{ nearly; or, } a=26.74=\text{about } 26\tfrac{3}{4} \text{ feet.}$$

[1] See footnote, page 94.

EXAMPLES. XXXVI.

1. $b = 19$ feet, $c = 20$ feet, $A = 60°$; find B and C.
2. $a = 376.375$ feet, $b = 251.765$ feet, $C = 78° 26'$; find B and A.
3. $a = .3$, $b = .363$, $C = 124° 56'$; solve the triangle.
4. $a = 135$, $b = 105$, $C = 60°$; find A.
5. $b = 8$, $c = 11$, $A = 93° 35'$; find a.
6. In a certain triangle two of the sides are 12 and 16 respectively; the angle included by them is 160.5°; find the other angles.

Case IV.

152. Given **two sides** and the **angle opposite** one of them, as b, c, B.

First, since $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$; $\therefore \sin C = \dfrac{c \sin B}{b}$.

C must be found from this equation.

When C is known, $A = 180° - B - C$,

and $$a = \frac{b \sin A}{\sin B}.$$

Which solves the triangle.

153. It should be noticed, however, that the angle C, found from the **trigonometrical** equation *sin C = a given quantity*, where C is an angle of a triangle, has **two** values, one less than 90°, and one greater than 90° (Art. 103).

The question arises, Are both these values admissible?

This may be decided as follows:

If B is not less than 90°, C must be less than 90°; and the smaller value for C **only** is admissible.

If B is less than 90° we proceed thus:

1. If b is less than $c \sin B$, then $\sin C$, which $= \dfrac{c \sin B}{b}$, is greater than 1. This is impossible. Therefore if b is less than $c \sin B$, there is **no solution** whatever.

2. If b is equal to $c \sin B$, then $\sin C = 1$, and therefore $C = 90°$; and there is only **one value** of C, viz. 90°.

3. If b is greater than $c \sin B$, and less than c, then B is less than C, and C may be obtuse or acute. In this case C may have either of the values found from the equation $\sin C = \dfrac{c \sin B}{b}$. Hence there are **two solutions**, and the triangle is said to be **ambiguous**.

4. If b is equal to or greater than c, then B is equal to or greater than C, so that C must be an acute angle; and the smaller value for C only is admissible.

154. The same results may be obtained geometrically.

Construction. Draw $AB = c$; make the angle $ABD =$ the given angle B; with centre A and radius $= b$ describe a circle; draw AD perpendicular to BD.

Then $AD = c \sin B$.

1. If b is less than $c \sin B$, *i.e.* less than AD, the circle will not cut BD at all, and the construction fails. (Fig. I.)

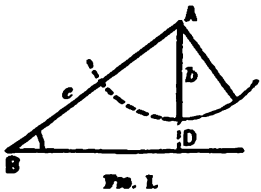


FIG. I.

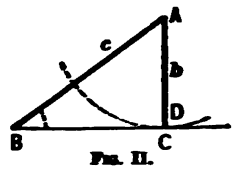


FIG. II.

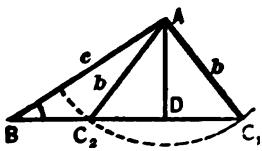


FIG. III.

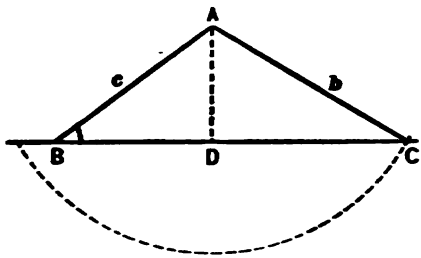


FIG. IV.

2. If b is equal to AD, the circle will *touch* the line BD in the point D, and the required triangle is the right-angled triangle ABD. (Fig. II.)

3. If b is greater than AD and less than AB, *i.e.* than c, the circle will cut the line BD in *two* points C_1, C_2, each on the same side of B. And we get two triangles ABC_1, ABC_2, each satisfying the given condition. (Fig. III.)

4. If b is equal to c, the circle cuts BD in B and in one other point C; if b is greater than c, the circle cuts BD in *two* points, but on *opposite* sides of B. In either case there is only one triangle satisfying the given condition. (Fig. IV.)

155. We may also obtain the same results algebraically from the formula $b^2 = c^2 + a^2 - 2\,ca \cos B$.

In this b, c, B are given, a is unknown. Write x for a, and we get the quadratic equation,

$$x^2 - 2\,c \cos B \cdot x = b^2 - c^2.$$

Whence, $x^2 - 2c\cos B \cdot x + c^2\cos^2 B = b^2 - c^2 + c^2\cos^2 B$

$$= b^2 - c^2\sin^2 B;$$

$$\therefore\ x = c\cos B \pm \sqrt{b^2 - c^2\sin^2 B}.$$

Let a_1, a_2 be the two values of x thus obtained; then

$$\left.\begin{aligned} a_1 &= c\cos B + \sqrt{b^2 - c^2\sin^2 B}; \\ a_2 &= c\cos B - \sqrt{b^2 - c^2\sin^2 B}. \end{aligned}\right\}$$

Which of these two solutions is admissible may be decided as follows:

1. When b is less than $c\sin B$, then $(b^2 - c^2\sin^2 B)$ is negative, so that a_1, a_2 are **impossible** quantities.

2. When b is equal to $c\sin B$, then $(b^2 - c^2\sin^2 B) = 0$, and $a_1 = a_2$; thus the *two* solutions become **one**.

3. When b is greater than $c\sin B$, then the two values a_1, a_2 are different and positive unless

$$\sqrt{b^2 - c^2\sin^2 B} \text{ is } > c\cos B;$$

i.e. unless $\quad b^2 - c^2\sin^2 B > c^2\cos^2 B,$

i.e. unless $\quad b^2 > c^2.$

4. When b is equal to c, then $a_2 = 0$; if b is greater than c, then a_2 is negative, and is therefore inadmissible. In either of these cases a_1 is the only available solution.

156. We give two examples. In the first there are two solutions; in the second there is only one.

EXAMPLE 1. Find A and C, having given that $b = 379.41$ chains, $c = 483.74$ chains, and $B = 34° 11'$.

$$\begin{aligned} \log\sin C &= \log c + \log\sin B - \log b \\ &= 2.68461 + 9.74961 - 2.57910 \\ &= 9.85511 = L\sin 45° 45'; \end{aligned}$$

$$\therefore\ C = 45° 45',\ \text{or}\ 180° - 45° 45' = 134° 15'.$$

Since b is less than c, each of these values is admissible.

When $C = 45° 45'$, then $A = 100° 4'$.

When $C = 134° 15'$, then $A = 11° 34'$.

EXAMPLE 2. Find A and C, when $b = 483.74$ chains, $c = 379.14$ chains, and $B = 34° 11'$.

$$\begin{aligned} \log\sin C &= \log c + \log\sin B - \log b \\ &= 2.5791 + 9.7496148 - 2.68461 \\ &= 9.64411 = \log\sin 26° 9'; \end{aligned}$$

$$\therefore\ C = 26° 9',\ \text{or}\ 180° - 26° 9' = 153° 51'.$$

Since b is greater than c, C must be less than 90°, and the larger value for C is inadmissible.

[It is also clear that $(153° 51' + 34° 11')$ is $> 180°$.]

$$\therefore\ C = 26° 9',\ A = 119° 40'.$$

EXAMPLES. XXXVII.

1. Discuss the following problems, using the method of Art. 153.

(*a*) $B = 45°$; $c = 12$; $b = \sqrt{50}$. (*b*) $B = 45°$; $c = 10$; $b = \sqrt{50}$.

(*c*) $B = 45°$; $c = \sqrt{40}$; $b = \sqrt{50}$. (*d*) $B = 45°$; $c = \sqrt{75}$; $b = \sqrt{50}$.

2. If $B = 40°$, $b = 140.5$ ft., $a = 170.6$ ft. Find A and C.

3. Find B, C, and c, having given that $A = 50°$, $b = 97$, $a = 119$. (See Ex. 2.)

4. If $C = 30°$, $b = 100$, $c = 45$, is the triangle ambiguous?

157. It is sometimes easier to use natural sines, cosines, and tangents (Art. 136), than to use processes of computation involving logarithms. The theory of the solution of triangles is the same in either case, and it is only a question as to which process involves the greater amount of labor, to perform the indicated arithmetical processes, or to use logarithms.

EXAMPLE. $a = 50$, $A = 78°$, $B = 27°$; find b.

We have $$b = a\frac{\sin B}{\sin A} = 50 \times \frac{\sin 27°}{\sin 78°} = \frac{50 \times 4540}{9781}.$$

$$\therefore\ b = 23.1.$$

MISCELLANEOUS EXAMPLES. XXXVIII.

1. Find A when $a = 374.5$, $b = 576.2$, $c = 759.3$ feet.
2. Find B when $a = 4001$, $b = 9760$, $c = 7942$ yards.
3. Find C when $a = 8761.2$, $b = 7643$, $c = 4693.8$ chains.
4. Find B when $A = 86° 19'$, $b = 4930$, $c = 5471$ chains.
5. Find C when $B = 32° 58'$, $c = 1873.5$, $a = 764.2$ chains.
6. Find c when $C = 108° 27'$, $a = 36541$, $b = 89170$ feet.
7. Find c when $B = 74° 10'$, $C = 62° 45'$, $b = 3720$ yards.
8. Find b when $B = 100° 19'$, $C = 44° 59'$, $a = 1000$ chains.
9. Find a when $B = 123° 7' 20''$, $C = 15° 9'$, $c = 9964$ yards.

Find the other two angles in the six following triangles:

10. $C = 100° 37'$, $b = 1450$, $c = 6374$ chains.

11. $C = 52° 10'$, $b = 643$, $c = 872$ chains.

12. $A = 76° 2' 30''$, $b = 1000$, $a = 2000$ chains.

13. $C = 54° 23'$, $b = 873.4$, $c = 752.8$ feet.

14. $C = 18° 21'$, $b = 674.5$, $c = 269.7$ chains.

15. $A = 29° 11' 43''$, $b = 7934$, $a = 4379$ feet.

16. The difference between the angles at the base of a triangle is $17° 48'$, and the sides subtending those angles are 105.25 feet and 76.75 feet; find the third angle.

17. If $b : c = 4 : 5$, $a = 1000$ yards and $A = 37° 19'$; find b.

The student will find some Examples of Solution of Triangles without the aid of logarithms, in Examples XLII.

158. *To find the* **Area** *of a Triangle.*

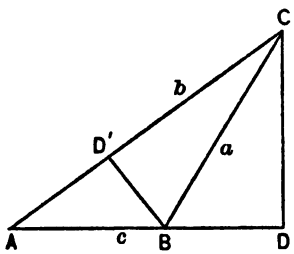


FIG. 54.

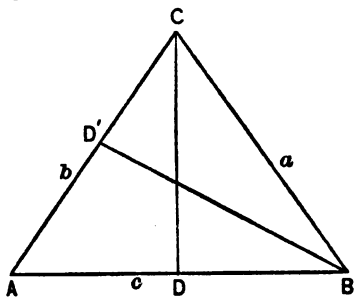


FIG. 55.

In either figure

Let Δ = area of triangle ABC.

Let CD be perpendicular to AB.

Let length of $CD = p$.

$$\Delta = \tfrac{1}{2} cp; \qquad \text{(Geom.)}$$

$$\therefore\ \Delta = \tfrac{1}{2} a \sin B \cdot c = \tfrac{1}{2} ac \sin B;$$

$$\text{or } \Delta = \tfrac{1}{2} b \sin A \cdot c = \tfrac{1}{2} bc \sin A.$$

Similarly, if BD' is perpendicular to AC,

$$\Delta = \tfrac{1}{2} ab \sin C.$$

In words:

The area of a triangle equals the continued product of any two sides and the sine of the angle included by them.

159. By i., Art. 115, $\sin A = \dfrac{2}{bc} \times \sqrt{s(s-a)(s-b)(s-c)}$;

$$\therefore\ \Delta = \sqrt{s(s-a)(s-b)(s-c)} = S. \qquad \text{(Art. 113.)}$$

160. Data, sufficient for the solution of a triangle, may be given in other terms than that of sides and angles, as in the four cases considered.

e.g. The triangle is determined when there are given area and two angles.

For, suppose in triangle ABC, we are given Δ, A, B.

$$ac \sin B = 2\Delta. \quad \therefore\ ac = \frac{2\Delta}{\sin B}.$$

$$a = \frac{c \sin A}{\sin C}. \quad \therefore\ c^2 = \frac{2\Delta \sin C}{\sin A \sin B}. \qquad \text{Q.E.D.}$$

See the following list of exercises for similar examples:

EXAMPLES. XXXIX.

1. Show that the triangle is determined when there are given: area, one angle, and side opposite the angle.

2. Find the area of the triangle ABC when

(i.) $a = 4$ feet, $b = 10$ feet, $C = 30°$.

(ii.) $b = 5$ inches, $c = 20$ inches, $A = 60°$.

(iii.) $c = 66\frac{2}{3}$ yards, $a = 15$ yards, $B = 17° 14'$ [$\sin 17° 14' = .29626$].

(iv.) $a = 13$, $b = 14$, $c = 15$.

(v.) $a = 10$ feet, the perpendicular from A on $BC = 20$ feet.

(vi.) $a = 625$, $b = 505$, $c = 904$.

CHAPTER XVI

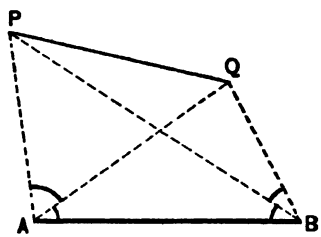
ON THE MEASUREMENT OF HEIGHTS AND DISTANCES

161. By the aid of the solution of triangles
we can find the distance between points which are inaccessible;
we can calculate the magnitude of angles which cannot be practically observed;
we can find the relative heights of distant and inaccessible points.

The method on which the trigonometrical survey of a country is conducted affords the following illustration:

162. *To find the distance between two distant objects.*

Two convenient positions A and B, on a level plain as far apart as possible, having been selected, the distance between A and B is



measured with the greatest possible care. This line AB is called the **base** line.

Next, the two distant objects, P and Q (church spires, for instance), visible from A and B, are chosen.

The angles PAB, PBA are observed. Then by Case II. Ch. XV., the lengths of the lines PA, PB are calculated.

Again, the angles QAB, QBA are observed; and by Case II. the lengths of QA and QB are calculated.

Thus the lengths of PA and QA are found.

The angle PAQ is observed; and then by Case III. the length of PQ is calculated.

Thus we are able to find not only the length PQ, but the angle which PQ makes with any line in the figure. The points P and Q are not necessarily accessible, the only condition being that P and Q must be *visible* from both A and B.

163. In practice, the points P and Q will generally be accessible, and then the line PQ, whose length has been calculated, may be used as a new base to find other distances.

164. *To find the height of a distant object above the point of observation.*

Let B be the point of observation; P the distant object. From B measure a base line BA of any convenient length, in any convenient direction; observe the angles PAB, PBA, and by Case II. Ch. XV.

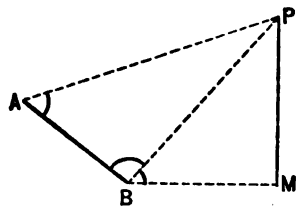


FIG. 56.

calculate the length of BP. Next observe at B the 'angle of elevation' of P; that is, the angle which the line BP makes with the horizontal line BM, M being the point in which the vertical line through P cuts the horizontal plane through B.

Then PM, which is the vertical height of P above B, can be calculated, for $PM = BP \cdot \sin MBP$.

EXAMPLE 1. *The distance between a church spire A and a milestone B is known to be 1764.3 feet; C is a distant spire. The angle CAB is $94° 54'$, and the angle CBA is $66° 39'$; find the distance of C from A.*

ABC is a triangle, and we know one side, c, and two angles (A and B), and therefore it can be solved by Case II. Ch. XV.

The angle $ACB = 180° - 94° 54' - 66° 39' = 18° 27'$.

Therefore the triangle is the same as that solved in Art. 148. Therefore $AC = 5118.2$ feet.

EXAMPLE 2. *If the spire C, in the last example, stands on a hill, and the angle of elevation of its highest point is observed at A to be $4° 19'$; find how much higher C is than A.*

The required height $x = AC \cdot \sin 4° 19'$ and AC is 5118.2 feet;

$$\therefore \log x = \log (AC \cdot \sin 4° 19')$$
$$= \log 5118.2 + \log \sin 4° 19' - 10$$
$$= 3.70911 + 8.87661$$
$$= 2.58573 = \log 385.24.$$
$$\therefore x = 385 \text{ ft. } 3 \text{ in. nearly.}$$

EXAMPLES. XL.

(EXAMPLES VII. CONSIST OF EASY EXAMPLES ON THIS SUBJECT.)

1. Two straight roads, inclined to one another at an angle of 60°, lead from a town A to two villages B and C; B on one road distant 30 miles from A, and C on the other road distant 15 miles from A. Find the distance from B to C. *Ans.* 25.98 m.

2. Two ships leave harbor together, one sailing N.E. at the rate of $7\frac{1}{2}$ miles an hour, and the other sailing north at the rate of 10 miles an hour. Prove that the distance between the ships after an hour and a half is 10.6 miles.

3. A and B are two consecutive milestones on a straight road, and C is a distant spire. The angles ABC and BAC are observed to be 120° and 45° respectively. Show that the distance of the spire from A is 3.346 miles.

4. If the spire C in the last question stands on a hill, and its angle of elevation at A is 15°, show that it is .896 of a mile higher than A.

5. If in Question (3) there is another spire D such that the angles DBA and DAB are 45° and 90° respectively and the angle DAC is 45°, prove that the distance from C to D is $2\frac{3}{4}$ miles very nearly.

6. A and B are two consecutive milestones on a straight road, and C is the chimney of a house visible from both A and B. The angles CAB and CBA are observed to be 36° 18′ and 120° 27′, respectively. Show that C is 2639.5 yards from B.

7. A and B are two points on opposite sides of a mountain, and C is a place visible from both A and B. It is ascertained that C is distant 1794 feet and 3140 feet from A and B, respectively, and the angle ACB is 58° 17′. Show that the angle which the line pointing from A to B makes with AC is 86° 55′ 49″.

8. A and B are two hill-tops 34920 feet apart, and C is the top of a distant hill. The angles CAB and CBA are observed to be 61° 53′ and 76° 49′, respectively. Prove that the distance from A to C is 51515 feet.

$\log 34920 = 4.54307$; $\log \sin 76° 49' = 9.98840$;
$\log 51515 = 4.71193$; $\log \operatorname{cosec} 41° 18' = 10.18045$.

9. From two stations A and B on shore, 3742 yards apart, a ship C is observed at sea. The angles BAC, ABC are simultaneously observed to be 72° 34′ and 81° 41′, respectively. Prove that the distance from A to the ship is 8522.7 yards.

$\log 3742 = 3.57310$; $\log \sin 81° 41' = 9.99540$;
$\log 8522.7 = 3.90057$; $\log \operatorname{cosec} 25° 45' = 10.36206$.

10. The distance between two mountain peaks is known to be 4970 yards, and the angle of elevation of one of them when seen from the other is $9° 14'$. How much higher is the first than the second? $\sin 9° 14' = .16045$.

Ans. 797.5 yards.

11. Two straight railways intersect at an angle of $60°$. From their point of intersection two trains start, one on each line, one at the rate of 40 miles an hour. Find the rate of the second train that at the end of an hour they may be 35 miles apart. *Ans.* Either 25 or 15 miles an hour. (Art. 153.)

12. A and B are two positions on opposite sides of a mountain; C is a point visible from A and B; AC and BC are 10 miles and 8 miles, respectively, and the angle BCA is $60°$. Prove that the distance between A and B is 9.165 miles.

13. A and B are consecutive milestones on a straight road; C is the top of a distant mountain. At A the angle CAB is observed to be $38° 19'$; at B the angle CBA is observed to be $132° 42'$, and the angle of elevation of C at B is $10° 15'$; show that the top of the mountain is 1243.5 yards higher than B.

$$\log \sin 38° 19' = 9.79239; \qquad \log 1760 = 3.24551;$$
$$\log \operatorname{cosec} 8° 59' = 10.80646; \qquad \log 1243.5 = 3.09465;$$
$$\log \sin 10° 15' = 9.25028.$$

14. A base line AB, 1000 feet long, is measured along the straight bank of a river; C is an object on the opposite bank; the angles BAC and CBA are observed to be $65° 37'$ and $53° 4'$ respectively; prove that the perpendicular breadth of the river at C is 829.87 feet.

15. A is the foot of a vertical pole, B and C are due east of A, and D is due south of C. The elevation of the pole at B is double that at C, and the angle subtended by AB at D is $\tan^{-1}\frac{1}{3}$. Also $BC = 20$ feet, $CD = 30$; find the height of the pole. — *Hobson's Trig.*

16. Two towers, one 200 feet high, the other 150 feet high, standing on a horizontal plane, subtend, at a point in the plane, angles of $30°$ and $60°$ respectively. The horizontal angle that their bases subtend at the same point is $120°$; how far are the two towers apart?

17. The diagonals of a parallelogram are in length d_1 and d_2, the angle between them is ϕ; show that the area of a parallelogram is $\frac{1}{2} d_1 d_2 \sin \phi$.

18. A man walking along a straight road at the rate of three miles an hour sees in front of him at an elevation of $60°$ a balloon which is travelling horizontally in the same direction at the rate of six miles an hour; ten minutes after he observes that the elevation is $30°$; prove that the height of the balloon above the road is $440\sqrt{3}$ yards.

19. A person standing at a point A, due south of a tower built on a horizontal plain, observes the altitude of the tower to be $60°$. He then walks to a point B due west from A and observes the altitude to be $45°$, and then at the point C in AB produced he observes the altitude to be $30°$; prove that $AB = BC$.

20. The angle of elevation of a balloon, which is ascending uniformly and vertically, when it is one mile high is observed to be $35° 20'$; 20 minutes later the elevation is observed to be $55° 40'$. How fast is the balloon moving?

Ans. $3(\sin 20° 20')(\sec 55° 40')(\operatorname{cosec} 35° 20')$ miles per hour.

21. A tower stands at the foot of an inclined plane whose inclination to the horizon is 9°; a line is measured up the incline from the foot of the tower of 100 feet in length. At the upper extremity of this line the tower subtends an angle of 54°; find the height of the tower. *Ans.* 114.4 feet.

22. The altitude of a certain rock is observed to be 47°, and after walking 1000 feet towards the rock, up a slope inclined at an angle of 32° to the horizon the observer finds that the altitude is 77°; prove that the vertical height of the rock above the first point of observation is 1034 feet. Sin 47° = .73135.

23. At the top of a chimney 150 feet high standing at one corner of a triangular yard, the angle subtended by the adjacent sides of the yard are 30° and 45° respectively; while that subtended by the opposite side is 30°; show that the lengths of the sides are 150 feet, 86.6 feet, and 106 feet respectively.

24. A flagstaff h feet stands on the top of a tower. From a point in the plane on which the tower stands, the angles of elevation of the top and bottom of the flagstaff are observed to be α and β respectively; prove that the height of the tower is $\dfrac{h \sin \beta \cos \alpha}{\sin (\alpha - \beta)}$ feet.

25. The angular elevation of the top of a steeple at a place due south of it is 45°, and at another place due west of the former station and distant a feet from it the elevation is 15°; show that the height of the steeple is $\frac{a}{2}(3^{\frac{1}{4}} - 3^{-\frac{1}{4}})$ feet.

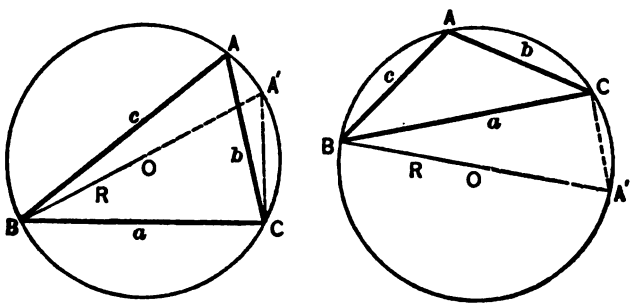
*CHAPTER XVII

MISCELLANEOUS THEOREMS

165. To find the radius of the **Circumscribing Circle.**

Let a circle $AA'CB$ be described about the triangle ABC. Let R stand for its radius. Let O be its centre. Join BO, and produce it to cut the circumference in A'. Join $A'C$.

Then, Fig. I., the angles BAC, $BA'C$ in the same segment are equal; Fig. II., the angles BAC, $BA'C$ are supplementary; also the angle $BC'A$ in a semicircle is a right angle.



I. Fig. 57. II.

Therefore, $\dfrac{CB}{A'B} = \sin CA'B = \sin CAB = \sin A,$

or, $\dfrac{a}{2R} = \sin A; \quad \therefore\ 2R = \dfrac{a}{\sin A}.$

166. Similarly, it may be proved that

$$2R = \frac{b}{\sin B}; \text{ and that } 2R = \frac{c}{\sin C}.$$

Hence, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$

Thus d, the value of each of these fractions, is the diameter of the circumscribing circle, which is another proof of the "law of sines," viz.

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

167. To find the radius of the **Inscribed Circle.**

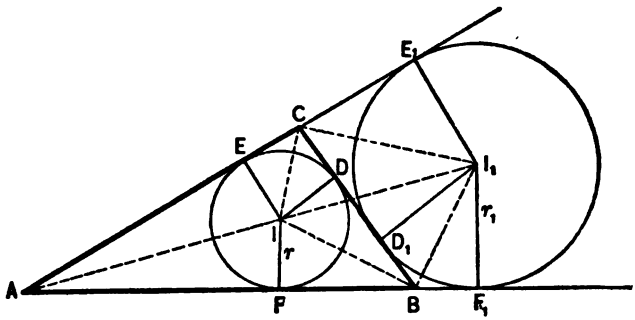


FIG. 58.

Let D, E, F be the points in which the circle inscribed in the triangle ABC touches the sides. Let I be the centre of the circle; let r be its radius. Then $ID = IE = IF = r$.

The area of the triangle ABC

$$= \text{area of } IBC + \text{area of } ICA + \text{area of } IAB.$$

And the area of the triangle $IBC = \frac{1}{2} ID \cdot BC = \frac{1}{2} r \cdot a$;

$$\therefore \text{ area of } ABC = \tfrac{1}{2} ID \cdot BC + \tfrac{1}{2} IE \cdot CA + \tfrac{1}{2} IF \cdot AB$$

$$= \tfrac{1}{2} ra + \tfrac{1}{2} rb + \tfrac{1}{2} rc;$$

or,
$$\Delta = \tfrac{1}{2} r(a+b+c) = \tfrac{1}{2} r \cdot 2s = rs.$$

$$\therefore r = \frac{\Delta}{s} = \frac{S}{s}.$$

168. A circle which touches one of the sides of a triangle and the other two sides produced is called an **Escribed Circle** of the triangle.

169. To find the radius of an **Escribed Circle.**

Let an escribed circle touch the side BC and the sides AC, AB produced in the points D_1, E_1, F_1 respectively. Let I_1 be its centre, r_1 its radius. Then

$$I_1D_1 = I_1E_1 = I_1F_1 = r_1.$$

The area of the triangle ABC

$$= \text{area of } ABI_1C - \text{area of } I_1BC,$$

$$= \text{area of } I_1CA + \text{area of } I_1AB - \text{area of } I_1BC;$$

or,
$$\Delta = \tfrac{1}{2} I_1E_1 \cdot CA + \tfrac{1}{2} I_1F_1 \cdot AB - \tfrac{1}{2} I_1D_1 \cdot BC,$$

$$= \tfrac{1}{2} r_1 b + \tfrac{1}{2} r_1 c - \tfrac{1}{2} r_1 a$$

$$= \tfrac{1}{2} r_1 (b + c - a) = \tfrac{1}{2} r_1 (2s - 2a) = r_1(s - a).$$

$$\therefore\ r_1 = \frac{\Delta}{s - a} = \frac{S}{s - a}.$$

Similarly if r_2 and r_3 be the radii of the other two escribed circles of the triangle ABC, then

$$r_2 = \frac{S}{s - b};\quad r_3 = \frac{S}{s - c}.$$

170. *To calculate the lengths AE, AE_1, AF_1.*

$$AE + EC + CD + DB + BF + FA = 2s. \quad \text{(Fig. 58, Art. 167.)}$$

But $AE = AF$,

$CD = EC$,

$DB = BF$ (tangents to the same circle from a given point).

$$\therefore\ AE + CD + BD = s, \text{ or } AE + a = s.$$

$$\therefore\ AE = s - a.$$

From similar triangles,

$$\frac{AE_1}{AE} = \frac{r_1}{r}.\quad \therefore\ AE_1 = \frac{AE \cdot r_1}{r} = \frac{(s - a)\Delta}{\frac{\Delta}{s}(s - a)} = s.$$

$$\therefore\ AE_1 = s = AF_1.$$

EXAMPLES. XLI.

1. Show (i.) $CD = EC = (s - c)$; (ii.) $BF = BD = s - b$; (iii.) $DD_1 =$ numerical difference of b and c.

2. Find the radii of the inscribed and each of the escribed circles of the triangle ABC when $a = 13$, $b = 14$, $c = 15$ feet.

3. Show that the triangles in which (i.) $a = 2$, $A = 60°$; (ii.) $b = \frac{2}{3} \cdot \sqrt{3}$, $B = 30°$ can be inscribed in the same circle.

4. Prove that $R = \dfrac{abc}{4S}$; find R in the triangle of (2).

5. Prove that if a series of triangles of equal perimeter are described about the same circle, they are equal in area.

6. If $A = 60°$, $a = \sqrt{3}$, $b = \sqrt{2}$, prove that the area $= \frac{1}{4}(3 + \sqrt{3})$.

7. Prove that each of the following expressions represents the area of the triangle ABC:

(i.) $\dfrac{abc}{4R}$.

(ii.) $2R^2 \sin A \cdot \sin B \cdot \sin C$.

(iii.) rs.

(iv.) $Rr(\sin A + \sin B + \sin C)$.

(v.) $\frac{1}{2} a^2 \sin B \cdot \sin C \cdot \operatorname{cosec} A$.

(vi.) $ra \operatorname{cosec} \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$.

(vii.) $(rr_1r_2r_3)^{\frac{1}{2}}$.

(viii.) $\frac{1}{2}(a^2 - b^2) \sin A \cdot \sin B \cdot \operatorname{cosec}(A - B)$.

Prove the following statements:

8. If a, b, c are in A.P., then $ac = 6rR$.

9. The area of the greatest triangle, two of whose sides are 50 and 60 feet, is 1500 sq. feet.

10. If the altitude of an isosceles triangle is equal to the base, R is five-eighths of the base.

171. We give here a geometrical proof of the following propositions.

PROP. I. *To prove that*

$$\cos A = 2\cos^2 \tfrac{1}{2} A - 1 = 1 - 2\sin^2 \tfrac{1}{2} A. \quad \text{(See Art. 91.)}$$

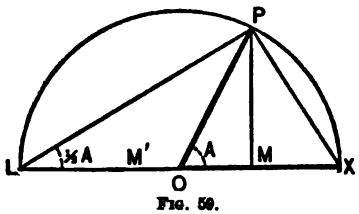


FIG. 59.

Let XOP be the angle A; with O as centre and any radius OX describe the semicircle XPL; join PL, PX, and draw PM perpendicular to LOX.

Then $$POM = OLP + OPL = 2\,OL$$

$$\therefore\ OLP = \tfrac{1}{2} POM = \tfrac{1}{2} A.$$

Now, $$\cos A = \frac{OM}{OP} = \frac{LM - LO}{OP} = \frac{2\,LM}{2\,OP} - \frac{OP}{OP}$$

$$= 2 \cdot \frac{LM}{LP} \cdot \frac{LP}{LX} - 1 = 2\cos OLP \cdot \cos OLP - 1$$

$$= 2\cos^2 \tfrac{1}{2} A - 1 \qquad \text{(i.)}$$

$$= 2(1 - \sin^2 \tfrac{1}{2} A) - 1$$

$$= 1 - 2\sin^2 \tfrac{1}{2} A. \qquad \text{(ii.)}$$

Prop. II. *To prove* $\sin A = 2 \sin \frac{1}{2} A \cdot \cos \frac{1}{2} A$,

$$\sin A = \frac{MP}{OP} = 2 \cdot \frac{MP}{LP} \cdot \frac{LP}{2\,OP} = 2 \cdot \frac{MP}{LP} \cdot \frac{LP}{LX}$$
$$= 2 \sin OLP \cdot \cos OLP = 2 \sin \tfrac{1}{2} A \cdot \cos \tfrac{1}{2} A.$$

(See Art. 91.)

Prop. III. *To prove that in any triangle*

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

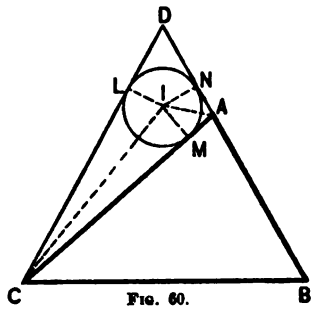


Fig. 60.

Let ABC be a triangle of which the angle B is greater than C.

Make the angle $BCD = B$ and produce BA to D.

In the triangle ACD inscribe the circle LMN, centre I, touching the sides in L, M, N; join IL, IM, IN, IA, IC.

Then $ICM = \frac{1}{2} LCM = \frac{1}{2}(DCB - ACB) = \frac{1}{2}(B - C)$,

$$IAM = \tfrac{1}{2} DAC = \tfrac{1}{2}(180^\circ - CAB) = (90^\circ - \tfrac{1}{2} A),$$
$$CM = CL = CD - LD = BD - ND = BN = BA + AM;$$
$$\therefore\ CM = \tfrac{1}{2}(CM + BA + AM) = \tfrac{1}{2}(AC + AB) = \tfrac{1}{2}(b + c),$$

and $$AM = AC - CM = b - \tfrac{1}{2}(b + c) = \tfrac{1}{2}(b - c).$$

Hence $$\frac{\tan \frac{B-C}{2}}{\cot \frac{A}{2}} = \frac{\tan ICM}{\tan (90^\circ - \frac{1}{2} A)} = \frac{\tan ICM}{\tan IAM}$$
$$= \frac{\frac{IM}{CM}}{\frac{IM}{AM}} = \frac{AM}{CM} = \frac{\frac{1}{2}(b-c)}{\frac{1}{2}(b+c)} = \frac{b-c}{b+c}.$$

Q.E.D.

172. If a line CF, drawn from the vertex C, of a triangle ABC, divides the vertical angle into two angles γ and γ', and the base into

two corresponding segments, m, m', and if CF makes the angle θ with the base, then $(m + m') \cos\theta = m \cot\gamma - m' \cot\gamma'$.

(Minchin's Statics, § 36.)

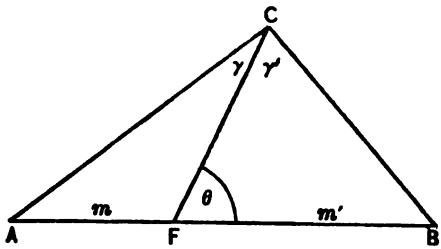


FIG. 61.

$$\frac{CF}{\sin A} = \frac{m}{\sin\gamma}; \quad \therefore\ CF = \frac{m \sin A}{\sin\gamma};$$

$$= \frac{m(\sin\theta\cos\gamma - \sin\gamma\cos\theta)}{\sin\gamma,};$$

also,
$$CF = \frac{m' \sin B}{\sin\gamma'} = \frac{m'(\sin\theta\cos\gamma' + \sin\gamma'\cos\theta)}{\sin\gamma'}.$$

Equate these values of CF.

$$m(\sin\theta\cot\gamma - \cos\theta) = m'(\sin\theta\cot\gamma' + \cos\theta);$$

$$\therefore\ (m + m')\cot\theta = m\cot\gamma - m'\cot\gamma'. \qquad (1) \text{ Q.E.D.}$$

COROLLARY. I. Suppose CF is a median; then

$$m = m',\ 2\cot\theta = \cot\gamma - \cot\gamma'. \qquad (2)$$

II. Suppose CF bisects angle C;

$$\frac{m + m'}{m - m'} = \frac{c}{m - m'} = \frac{\cot\frac{C}{2}}{\cot\theta}. \qquad (3)$$

III. Suppose CF is perpendicular to the base; then

$$\cot\theta = 0,\ m'\cot\gamma' = m\cot\gamma. \qquad (4)$$

This is evident geometrically.

EXERCISE. Show that

$$(m + m')\cot\theta = m'\cot A - m\cot B. \qquad (5)$$

173. Let ABC be any triangle, and AD, BE, CF the medians drawn from A, B, C respectively. Let AD divide the angles A into two angles α, α'; BE divide angle B into β, β', and CF divide angle C into γ, γ'.

From (2), $\qquad 2\cot\theta = \cot\gamma - \cot\gamma'$.

From (5), $\qquad 2\cot\theta = \cot A - \cot B$;

$$\therefore\ \cot A - \cot B = \cot\gamma - \cot\gamma'.$$

Similarly, $$\cot B - \cot C = \cot \alpha - \cot \alpha';$$
$$\cot C - \cot A = \cot \beta - \cot \beta'.$$

Adding these equations, we obtain

$$\cot \alpha + \cot \beta + \cot \gamma = \cot \alpha' + \cot \beta' + \cot \gamma'.$$

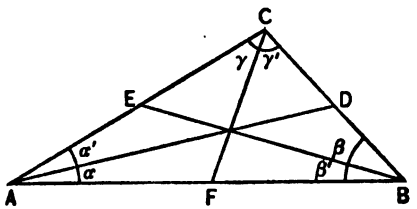


FIG. 62.

EXAMPLES. XLII.

MISCELLANEOUS EXERCISES.

1. Define the terms sine, cotangent; and prove that if A be *any* angle,

$$\sin^2 A + \cos^2 A = 1.$$

If $\tan A = \frac{3}{4}$, find $\sin A$ and $\cos A$.

2. Find the sine, cosine, and tangent of 30°.

In a triangle ABC, the angle C is a right angle, the angle A is 60°, and the length of the perpendicular let fall from C on AB is 20 feet; find the length of AB.

3. Prove geometrically that $\cos(180° - A) = -\cos A$.

4. Prove (1) $\sin(A + B) \cdot \sin(A - B) = \sin^2 A - \sin^2 B$;

(2) $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}.$$

5. Prove that

$$\cos^2 A - \cos A \cos(60° + A) + \sin^2(30° - A) = \tfrac{3}{4}.$$

6. Find the greatest side of the triangle of which one side is 2183 feet and the adjacent angles are 78° 14′ and 71° 24′.

---

7. Express the other trigonometrical ratios in terms of the cosine.

8. Prove $$\sin(180° + A) = -\sin A;$$
$$\tan(90° + A) = -\cot A.$$

9. Write down the sines of all the angles which are multiples of 30° and less than 360°.

10. Prove $$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$$

11. If $\tan A + \sec A = 2$, prove that $\sin A = \frac{3}{5}$, when A is less than 90°.

If $\sin A = \frac{4}{5}$, prove that $\tan A + \sec A = 3$, when A is less than 90°.

12. The length of the greatest side of a triangle is 1035.43 feet, and the three angles are 44°, 66°, and 70°. Solve the triangle.

13. Express the other trigonometrical ratios in terms of the cotangent.

14. Prove that $\operatorname{cosec}(180° + A) = -\operatorname{cosec} A$.

15. Write down the tangents of all the angles which are multiples of 30° and less than 360°.

16. If $\tan A + \sec A = 3$, prove that $\sin A = \frac{4}{5}$, when A is less than 90°.

If $\sin A = \frac{3}{5}$, prove that $\tan A + \sec A = 2$, when A is less than 90°.

17. Find the sines of the three angles of the triangle whose sides are 193, 194, and 195 feet.

18. Investigate the following formulæ:

(1) $\cos \frac{3A}{2} = (2 \cos A - 1) \cos \frac{1}{2} A$;

(2) $\cos \theta - \cos(\theta + \delta) = \sin \theta \sin \delta (1 + \cot \theta \tan \frac{1}{2} \delta)$.

---

19. Define the secant of an angle.

Prove the formula $\dfrac{1}{\sec^2 A} + \dfrac{1}{\operatorname{cosec}^2 A} = 1$.

If $\sin A = \frac{1}{3}$, find $\sec A$.

20. Find the logarithms of $\sqrt{(32)}$; and of .03125 to the base $\sqrt[3]{2}$.

21. Express the sine, cosine, and tangent of each of the angles 1962°, 2376°, 2844°, in terms of the trigonometrical functions of angles lying between 0 and 45°.

22. Prove the formula to express the cosine of the sum of two angles in terms of the sines and cosines of those angles.

Express $\cos 5\alpha$ in terms of $\cos \alpha$.

23. Find solutions of the equations

(i.) $\sec \theta \operatorname{cosec} \theta - \cot \theta = \sqrt{3}$;

(ii.) $\sin 2\theta - \sin \theta = \cos 2\theta + \cos \theta$.

24. A ring 10 inches in diameter is suspended from a point 1 foot above its centre by six equal strings attached to its circumference at equal intervals; find the cosine of the angle between two consecutive strings.

---

25. Define 1°. Assuming that $\frac{22}{7}$ is the circular measure of two right angles, express the angle $A°$ in circular measure.

Find the number of degrees in the angle whose circular measure is .1.

26. Find the trigonometrical ratios of the angle whose cosine is $\frac{4}{5}$.

27. Prove that (1) $\cos(180° + A) = \cos(180° - A)$;

(2) $\tan(90° + A) = \cot(180° - A)$.

28. Prove $$\sin x(2\cos x - 1) = 2\sin\frac{x}{2}\cos\frac{3x}{2}.$$

29. Express $\log_{10} 5.832$, $\log_{10}\sqrt[3]{(35)}$, and $\log_{10} .304\dot{8}$ in terms of $\log_{10} 2$, $\log_{10} 3$, $\log_{10} 7$.

30. If the angle opposite the side a be $60°$, and if b, c be the remaining sides of the triangle, prove that

$$(a + b + c)(b + c - a) = 3bc.$$

---

31. Assuming $\frac{22}{7}$ to be the circular measure of two right angles, express in degrees the angle whose circular measure is θ. Find the number of degrees in an angle whose circular measure is $\frac{1}{7}$.

32. Show from the definitions of the trigonometrical function that

$$\sin^2 A + \cot^2 A + \cos^2 A = \operatorname{cosec}^2 A.$$

Prove that $$\frac{\tan A + \sec A + 1}{\tan A + \sec A - 1} = \frac{\sec A + 1}{\tan A}.$$

33. Prove $$\sin x(2\cos x + 1) = 2\cos\frac{x}{2}\sin\frac{3x}{2}.$$

34. Find the logarithms of $\sqrt{(27)}$ and $.0\dot{3}\dot{7}$ to the base $\sqrt[3]{3}$.

35. If $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3\sin A \sin B$, and $A + B + C = 180°$, prove that $C = 60°$.

36. Given $A = 18°$, $B = 144°$, and $b = 1$; solve the triangle.

---

37. Give the trigonometrical definition of an angle.

What angle does the minute-hand of a clock describe between twelve o'clock and 20 minutes to four?

38. Express the cosine and the tangent of an angle in terms of the sine.

The angle A is greater than $90°$, but less than $180°$, and $\sin A = \frac{1}{3}$; find $\cos A$.

39. Find all the values of θ between 0 and 2π for which $\cos\theta + \cos 2\theta = 0$.

40. If in a triangle $a\cos A = b\cos B$, the triangle will be either isosceles or right-angled.

41. The sides are 1 foot and $\sqrt{3}$ feet respectively, and the angle opposite to the shorter side is $30°$; solve the triangle.

42. The sides of a triangle are 2, 3, 4; find the greatest angle, having given

$$\log 2 = .30103,$$
$$\log 3 = .47712,$$
$$\log\tan 52° 15' = .11110,$$
$$\log\tan 52° 14' = .11083.$$

---

43. Distinguish between Euclid's definition of an angle and the trigonometrical definition.

What angle does the minute-hand of a clock describe between half-past four and a quarter-past six?

44. Express the sine and the cosine of an angle in terms of the tangent.

The angle A is greater than 180°, but less than 270°, and $\tan A = \frac{1}{2}$. Find $\sin A$.

45. Prove (i.) $$\sin 2A = \frac{2 \cot A}{1 + \cot^2 A}.$$

(ii.) Show that if $$A + B + C = 90°,$$
$$\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C.$$

46. Find all the values of θ between 0 and 2π, for which
$$\sin \theta + \sin 2\theta = 0.$$

47. If in a triangle $b \cos A = a \cos B$, show that the triangle is isosceles.

48. The sides are 1 foot and $\sqrt{2}$ feet respectively, and the angle opposite to the shorter side is 30°. Solve the triangle.

---

49. Express in degrees, minutes, and seconds (1) the angle whose circular measure is $\frac{1}{20}\pi$; (2) the angle whose circular measure is 5.

If the angle subtended at the centre of a circle by the side of a regular heptagon be the unit of angular measurement, by what number is an angle of 45° represented?

50. Prove that
$$(\sin 30° + \cos 30°)(\sin 120° + \cos 120°) = \sin 30°.$$

51. Prove the formulæ:

(1) $$\cos^2(\alpha + \beta) - \sin^2 \alpha = \cos \beta \cos(2\alpha + \beta);$$

(2) $$1 + \cot \alpha \cot \tfrac{1}{2}\alpha = \operatorname{cosec} \alpha \cot \tfrac{1}{2}\alpha.$$

52. Find solutions of the equations:

(1) $5 \tan^2 x - \sec^2 x = 11$; (2) $\sin 5\theta - \sin 3\theta = \sqrt{2} \cdot \cos 4\theta$.

53. Two sides of a triangle are 10 feet and 15 feet in length, and the angle between them is 30°. What is its area?

54. Given that
$$\sin 40° 29' = 0.64922, \quad \sin 40° 30' = 0.64944;$$
find the angle whose sine is 0.64930.

---

55. Express in circular measure (1) 10′; (2) $\frac{1}{3}$ of a right angle.

If the angle subtended at the centre of a circle by the side of a regular pentagon be the unit of angular measurement, by what number is a right angle represented?

56. If $\sec \alpha = 7$, find $\tan \alpha$ and $\operatorname{cosec} \alpha$.

57. Prove the formulæ:

(1) $$\cos^2(\alpha - \beta) - \sin^2(\alpha + \beta) = \cos 2\alpha \cos 2\beta;$$

(2) $$1 + \tan \alpha \tan \tfrac{1}{2}\alpha = \sec \alpha.$$

58. Find solutions of the equations:

(1) $5 \tan^2 x + \sec^2 x = 7$; (2) $\cos 5\theta + \cos 3\theta = \sqrt{2} \cdot \cos 4\theta$.

59. The lengths of the sides of a triangle are 3 feet, 5 feet, and 6 feet. What is its area?

60. Given that

$$\sin 38^\circ 25' = 0.62137\,; \quad \sin 38^\circ 26' = 0.62160\,;$$

find the angle whose sine is (0.62150).

---

61. Which is greater, 76° or 1.2^c?

62. Determine geometrically $\cos 30^\circ$ and $\cos 45^\circ$.

If $\sin A$ be the arithmetic mean between $\sin B$ and $\cos B$, then

$$\cos 2A = \cos^2 (B + 45^\circ).$$

63. Establish the following relations:

(1) $$\tan^2 A - \sin^2 A = \tan^2 A \sin^2 A\,;$$

(2) $$\cot A - \cot 2A = \operatorname{cosec} 2A\,;$$

(3) $$\frac{\sin (x + 3y) + \sin (3x + y)}{\sin 2x + \sin 2y} = 2 \cos (x + y).$$

64. Express $\log_{10} \sqrt{(28)}$, $\log_{10} 3.888$, $\log_{10} .174\dot{2}$ in terms of $\log_{10} 3$, $\log_{10} 5$, $\log_{10} 7$.

65. Prove that $\sin (A + B) = \sin A \cos B + \cos A \sin B$, and deduce the expression for $\cos (A + B)$.

Show that

$$\sin A \cos (B + C) - \sin B \cos (A + C) = \sin (A - B) \cos C.$$

66. One side of a triangular lawn is 102 feet long, its inclinations to the other sides being $70^\circ 30'$, $78^\circ 10'$, respectively. Determine the other sides and the area. $\log \sin 70^\circ 30' = 9.974$, $\log 102 = 2.009$, $\log \sin 78^\circ 10' = 9.990$, $\log 185 = 2.267$, $\log \sin 31^\circ 20' = 9.716$, $\log 192 = 2.283$, $\log 2 = .301$, $\log 9234 = 3.965$.

---

67. Which is greater, 126° or the angle whose circular measure is 2.3?

68. Establish the following relations:

(1) $$\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A\,;$$

(2) $$\tan A + \cot 2A = \operatorname{cosec} 2A\,;$$

(3) $$\frac{\cos (x - 3y) - \cos (3x - y)}{\sin 2x + \sin 2y} = 2 \sin (x - y).$$

69. Given $\log_{10} 2 = .3010300$, $\log_{10} 9 = .9542425$; find without using tables, $\log_{10} 5$, $\log_{10} 6$, $\log_{10} .0216$, and $\log_{10} \sqrt[5]{(.375)}$.

70. Prove that $\sin 30^\circ + \sin 120^\circ = \sqrt{2} \cos 15^\circ$.

71. Establish the identities:

(1) $1 + \cos A + \sin A = \sqrt{2 (1 + \cos A)(1 + \sin A)}$;

(2) $\operatorname{cosec} 2A = \dfrac{\operatorname{cosec}^2 A}{2\sqrt{\operatorname{cosec}^2 A - 1}}$;

(3) $\sin \dfrac{2\pi}{7} + \sin \dfrac{4\pi}{7} - \sin \dfrac{6\pi}{7} = 4 \sin \dfrac{\pi}{7} \sin \dfrac{3\pi}{7} \sin \dfrac{5\pi}{7}$.

72. The sides of a triangular lawn are 102, 185, and 192 feet in length, the smallest angle being approximately 31° 20′. Find its other angles and its area.

$$\log 102 = 2.009, \qquad \log \sin 31° 20' = 9.716,$$
$$\log 185 = 2.267, \qquad \log \sin 70° 30' = 9.974,$$
$$\log 192 = 2.283, \qquad \log \sin 78° 10' = 9.990,$$
$$\log 2 = .301, \log 9234 = 3.965.$$

---

73. If the circumference of a circle be divided into five parts in arithmetical progression, the greatest part being six times the least, express in radians the angle each subtends at the centre.

74. Define the sine of an angle, wording your definition so as to include angles of any magnitude.

Prove that $\sin(90° + A) = \cos A$,

and $\cos(90° + A) = -\sin A$,

and by means of these deduce the formulæ

$$\sin(180° + A) = -\sin A, \ \cos(180° + A) = -\cos A.$$

75. Prove the formulæ:

(1) $\cot^2 A = \operatorname{cosec}^2 A - 1$;

(2) $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A$.

Verify (2) when $A = 30°$.

76. Evaluate to 4 significant figures by the aid of the table of logarithms

$$\frac{7.891}{.0345} \times \sqrt[7]{.(008931)}.$$

77. If $\sin B$ be the geometric mean between $\sin A$ and $\cos A$, then $\cos 2B = 2\cos^2(A + 45°)$.

78. The lengths of two of the sides of a triangle are 1 foot and $\sqrt{2}$ feet respectively, the angle opposite the shorter side is 30°. Prove that there are two triangles which satisfy these conditions; find their angles, and show that their areas are in the ratio $\sqrt{3} + 1 : \sqrt{3} - 1$.

---

79. If the circumference of a circle be divided into six parts in arithmetical progression, the greatest being six times the least, express in radians the angle each subtends at the centre.

80. Define the tangent of an angle, wording your definition so as to include angles of any magnitude.

Prove that $\tan(90° + A) = -\cot A$, and by means of this formula deduce the formula $\tan(180° + A) = \tan A$.

81. Compute by means of tables the value of

$$\frac{6.12}{.4131} \times \sqrt[5]{54.17}.$$

82. Prove that $\cos(A + B) = \cos A \cos B - \sin A \sin B$, and deduce the expression for $\sin(A + B)$.

Show that $\cos A \cos(B + C) - \cos B \cos(A + C) = \sin(A - B) \sin C$.

83. Establish the identities:

(1) $1 + \cos A - \sin A = \sqrt{2(1 + \cos A)(1 - \sin A)}$;

(2) $\sec 2A = \dfrac{\sec^2 A}{2 - \sec^2 A}$;

(3) $\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7} + 4\cos\dfrac{\pi}{7}\cos\dfrac{3\pi}{7}\cos\dfrac{5\pi}{7} + 1 = 0.$

84. Two adjacent sides of a parallelogram 5 in. and 3 in. long respectively, include an angle of 60°. Find the lengths of the two diagonals and the area of the figure.

---

85. Investigate the following formulæ:

(1) $\sin\dfrac{3A}{2} = (1 + 2\cos A)\sin\frac{1}{2}A$;

(2) $\sin(\theta + \delta) - \sin\theta = \cos\theta\sin\delta\,(1 - \tan\theta\tan\frac{1}{2}\delta)$.

86. Prove that

(1) $\sin 10° + \sin 50° = \sin 70°$;

(2) $\sqrt{3} + \tan 40° + \tan 80° = \sqrt{3}\tan 40°\tan 80°$;

(3) if $A + B + C = 180°$,

$$\frac{\sin A - \sin B\cos C}{\cos B} = \frac{\sin B - \sin A\cos C}{\cos A}.$$

87. Prove by means of the logarithmic table that

$$\frac{1}{73^{-\frac{1}{7}}} = 1.846 \text{ nearly.}$$

88. The length of one side of a triangle is 1006.62 feet and the adjacent angles are 44° and 70°. Solve the triangle, having given

$L\sin 44° = 9.8417713$, $\quad L\sin 70° = 9.9729858$,

$L\sin 66° = 9.9607302$, $\quad \log 1006.62 = 3.0028656$,

$\log 7654321 = 6.8839067$, $\quad \log 103543 = 5.0151212$.

---

89. Find the length of the arc of a circle whose radius is 8 feet which subtends at the centre an angle of 50°, having given

$$\pi = 3.1416.$$

90. Prove that $\sin A = -\sin(A - 180°)$.

Find the sines of 30° and 2010°.

91. Given that the integral part of $(3.1622)^{100000}$ contains fifty thousand digits, find $\log_{10} 31622$ to five places of decimals.

92. Prove that

(1) $\cos^2 A + \cos^2 B - 2\cos A\cos B\cos(A + B) = \sin^2(A + B)$;

(2) $\cos^2 A + \sin^2 A\cos 2B = \cos^2 B + \sin^2 B\cos 2A$.

93. Prove that in any triangle

$$a^2\cos 2B + b^2\cos 2A = a^2 + b^2 - 4ab\sin A\sin B.$$

94. If $a = 123$, $B + 29° 17'$, $C = 135°$, find c, having given

$\log 123 = 2.0899051$, $\quad \log 2 = .3010300$,

$\log 3211 = 4.5066403$, $\quad$ diff. for 1 = 1352,

$\log\sin 15° 43' = 9.4327777$.

CHAPTER XVIII

RELATIONS AMONG THE SIDES AND THE ANGLES OF A SPHERICAL TRIANGLE

174. The succeeding pages contain a brief discussion of some of the properties of spherical triangles.

For the sake of ready reference, we shall enumerate, without proof, some properties of solid figures. The statements contained in sections 175–177 are proved in works on solid geometry, to which the student is referred.

175. Definitions and Theorems. The curve of intersection of a plane and a sphere is a circle.

When the plane of the circle passes through the centre of the sphere, their curve of intersection is called a **great circle.**

One great circle can be passed through any two points on the surface of the sphere, and only one if these points are not extremities of a diameter of the sphere.

A spherical figure is any part of the surface of the sphere bounded by arcs of great circles.

A **spherical polygon** is a spherical figure bounded by more than two arcs. The arcs are called the **sides** of the polygon. The intersections of the arcs are called the **vertices** of the polygon.

176. The **angle** between two great circles is measured by the angle between the tangents drawn to the circles at their point of intersection. This is called a **spherical angle.**

The angle between two great circles equals the angle between their planes.

177. Spherical Triangles. A **spherical triangle** is a spherical polygon of three sides.

Let ABC be a spherical triangle.

Let O be the centre of the sphere.

By the letters A, B, C we shall indicate *geometrically* the three angular points of the triangle ABC; *algebraically*, the three angles at those points respectively. By the letters a, b, c we shall indicate the measures of the sides opposite A, B, C, respectively.

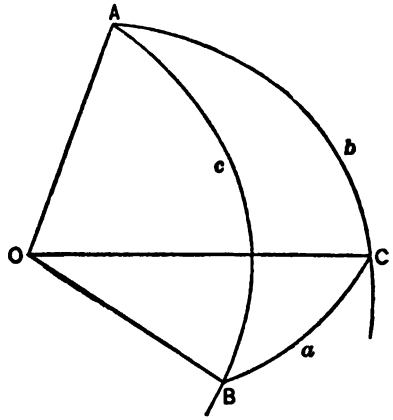


FIG. 63.

a, b, c are measured by the angles at the centre of the sphere, and hence they are measured in angular units; *e.g.* c is measured by angle AOB.

We know, then, the following properties:

1. *The sum of two sides of a triangle is greater than the third.
2. The greatest side is opposite the greatest angle, and conversely.
3. Any angle $A < 180°$.
4. $(A + B + C) < 540°$ and $> 180°$.
5. Any side $a < 180°$.
6. $(a + b + c) < 360°$.
7. $A - B$ and $a - b$ are of the same sign.
8. A side which differs from 90° more than another side is in the same quadrant as the angle opposite it.
9. If $A'B'C'$ is the polar triangle of ABC, and if A', B', C' are its angles, and a', b', c' the corresponding sides, then

$$A = 180° - a'; \quad A' = 180° - a;$$
$$B = 180° - b'; \quad B' = 180° - b;$$
$$C = 180° - c'; \quad C' = 180° - c.$$

* NOTE. We know that in general three great circles intersect in such a way as to form eight spherical triangles; and that at least one of these triangles satisfies the conditions of Art. 177. We shall consider only such triangles.

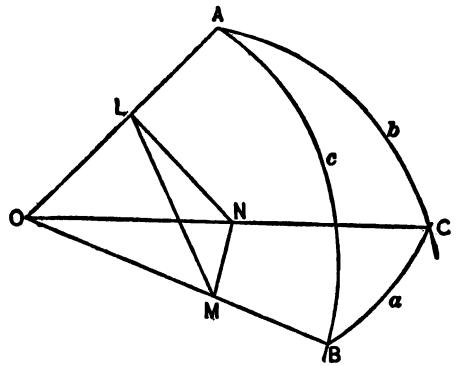


FIG. 64.

178. *To prove* $\cos a = \cos c \cos b + \sin b \sin c \cos A$.

Let ABC be a spherical triangle.

Let O be the centre of the sphere, and $OA = OB = OC =$ radius of the sphere.

In the plane AOB, draw LM perpendicular to OA.

In the plane AOC, draw LN perpendicular to OA.

Then the plane angle $MLN = A$ (Art. 176).

In the plane triangles LMN and MON,

$$\overline{MN}^2 = \overline{LM}^2 + \overline{LN}^2 - 2\,\overline{LM}\;\overline{LN}\cos A. \qquad \text{(Art. 107.)}$$

$$\overline{MN}^2 = \overline{OM}^2 + \overline{ON}^2 - 2\,\overline{OM}\cdot\overline{ON}\cos a.$$

Equate these values of $\overline{MN}^2$,

$$\overline{OM}^2 - \overline{LM}^2 + \overline{ON}^2 - \overline{LN}^2 - 2\,\overline{OM}\cdot\overline{ON}\cos a + 2\,LM\cdot LN\cos A = 0.$$

But $\qquad \overline{OM}^2 - \overline{LM}^2 = \overline{OL}^2$, and $\overline{ON}^2 - \overline{LN}^2 = \overline{OL}^2$;

$$\therefore\ 2\,\overline{OL}^2 - 2\,\overline{OM}\cdot\overline{ON}\cos a + 2\,LM\cdot LN\cos A = 0;$$

or $$\frac{OL}{OM}\cdot\frac{OL}{ON} + \frac{LM}{OM}\cdot\frac{LN}{ON}\cos A = \cos a;$$

i.e. $$\cos c\cos b + \sin b\sin c\cos A = \cos a. \qquad (a)$$

179. By reference to Fig. 63 it is evident that the demonstration of the preceding article requires that c and b are each less than 90°, while a is unrestricted.

Suppose, now, that one of the sides, b, say, is greater than 90°. Produce the great circles AC and BC. They intersect again in C', Fig. 65.

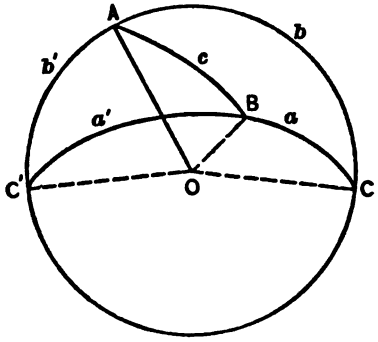


FIG. 65.

Consider now the triangle ABC.

$$AB < 90^\circ \text{ (hypothesis).}$$

$$AC' = (180^\circ - AC) < 90^\circ.$$

We can now apply the formula (a), of the preceding article, to ABC'. Hence

$$\cos a' = \cos b' \cos c + \sin b' \sin c \cos (180^\circ - A). \qquad (b)$$

But $\qquad \cos a' = -\cos a; \qquad$ (Art. 59.)

and $\qquad \cos b' = -\cos b; \qquad$ (Art. 59.)

and $\qquad \cos (180^\circ - A) = -\cos A.$

Substituting these values in (b) we obtain

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Similarly, if both b and c are greater than 90°, it may be shown that

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

So that (a) is true for all spherical triangles which we are considering. Similarly we can express the other angles in terms of the sides. We therefore have this relation involving the sides and one angle.

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A, \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B, \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C. \end{aligned}\right\} \qquad (1)$$

Whence

$$\left.\begin{aligned} \cos A &= \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \\ \cos B &= \frac{\cos b - \cos c \cos a}{\sin c \sin a}, \\ \cos C &= \frac{\cos c - \cos a \cos b}{\sin a \sin b}. \end{aligned}\right\} \qquad (2)$$

The last two formulæ of either set can be derived from the first one of that set by making a cyclical interchange of a into b, b into c, c into a, and at the same time changing A into B, B into C, C into A.

180. *To show that*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \tag{3}$$

This is a relation involving the sides and the opposite angles.

PROOF. $\sin^2 A = 1 - \cos^2 A$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}. \qquad \text{(Art. 179.)}$$

Substitute for $\sin^2 b$, $1 - \cos^2 b$, and $\sin^2 c = 1 - \cos^2 c$, and reducing we obtain

$$\sin^2 A = \frac{1 - \cos^2 b - \cos^2 c - \cos^2 a + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Multiplying both sides of the equation by $\frac{1}{\sin^2 a}$, and extracting the square root of each side, we obtain

$$\frac{\sin A}{\sin a} = \frac{\sqrt{1 - \cos^2 b - \cos^2 c - \cos^2 a + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}.$$

In a similar way we might solve for $\frac{\sin B}{\sin b}$. But this is *equivalent* to a *cyclical interchange* of the letters as described in Art. 179. But an interchange of the letters changes the left side of the equation into $\frac{\sin B}{\sin b}$, and leaves the right side unchanged. Hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$= \frac{\sqrt{1 - \cos^2 b - \cos^2 c - \cos^2 a + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}, \qquad \text{Q.E.D.}$$

since $A < 180°$, and since $a < 180°$ (Art. 177).

Therefore $\sin A$ and $\sin a$ are positive. Therefore there is no *ambiguity* of sign.

181. We shall add a geometrical proof of the theorem.

Take L any point in OA. From L let a perpendicular fall on plane COB, piercing it in G; from G draw GM and GN perpe-

dicular to OB and OC respectively. Join L and M; join L and N. Then $LM \perp OB$, and $LN \perp OC$. (Geom.)

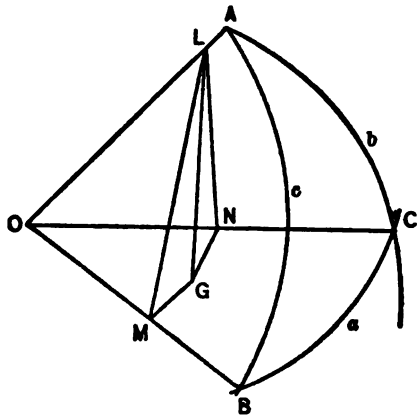


FIG. 66.

Angle B is measured by angle LMG.
Angle C is measured by angle LNG.

$$LM \cdot \sin B = LG = LN \sin C;$$

$$LM = OL \sin AOB = OL \sin c;$$

$$LN = OL \sin AOC = OL \sin b;$$

$$\therefore\ OL \sin C \sin B = OL \sin b \sin c.$$

$$\therefore\ \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

182. If $A'B'C'$ be the polar triangle of ABC, then (Fig. 67)

$$\left.\begin{aligned} a' &= 180° - A; \\ b' &= 180° - B; \\ c' &= 180° - C. \end{aligned}\right\} \qquad \text{(Art. 177.)}$$

We may now apply formula (1) to $A'B'C'$.

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'. \qquad (a)$$

Now,
$$\cos a' = -\cos A, \quad \sin b' = \sin B;$$
$$\cos b' = -\cos B, \quad \sin c' = \sin C;$$
$$\cos c' = -\cos C, \quad \cos A' = -\cos a.$$

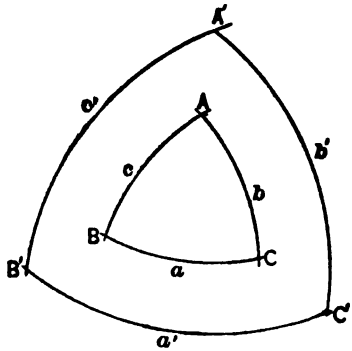


FIG. 67.

Substituting these values in (a), we obtain

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a.$$

Treating the other two formulæ similarly, we obtain

$$\left.\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a; \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b; \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c. \end{aligned}\right\} \qquad (4)$$

Relation involving the angles and one side.

Solving these equations for $\cos a$, $\cos b$, $\cos c$, we have

$$\left.\begin{aligned} \cos a &= \frac{\cos A + \cos B \cos C}{\sin B \sin C}; \\ \cos b &= \frac{\cos B + \cos C \cos A}{\sin C \sin A}; \\ \cos c &= \frac{\cos C + \cos A \cos B}{\sin A \sin B}. \end{aligned}\right\} \qquad (5)$$

183. From (1),

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Substituting in this formula the value of $\cos c$ obtained from (1), we obtain

$$\cos a (1 - \cos^2 b) = \sin a \sin b \cos b \cos C + \sin b \sin c \cos A;$$

or

$$\cos a \sin b = \sin a \cos b \cos C + \sin c \cos A;$$

and similarly,

$$\left.\begin{aligned} \cos b \sin a &= \sin b \cos a \cos C + \sin c \cos B; \\ \cos b \sin c &= \sin b \cos c \cos A + \sin a \cos B; \\ \cos c \sin b &= \sin c \cos b \cos A + \sin a \cos C; \\ \cos c \sin a &= \sin c \cos a \cos B + \sin b \cos C; \\ \cos a \sin c &= \sin a \cos c \cos B + \sin b \cos A. \end{aligned}\right\} \qquad (6)$$

(6) is a **relation involving two angles and the sides.**

By treating (3) similarly,

$$\left.\begin{aligned}
\cos A \sin B &= \cos a \sin C - \cos c \cos B \sin A;\\
\cos C \sin B &= \cos c \sin A - \cos a \cos B \sin C;\\
\cos C \sin A &= \cos c \sin B - \cos b \cos A \sin C;\\
\cos B \sin A &= \cos b \sin C - \cos c \cos A \sin B;\\
\cos B \sin C &= \cos b \sin A - \cos a \cos C \sin B;\\
\cos A \sin C &= \cos a \sin B - \cos b \cos C \sin A.
\end{aligned}\right\} \qquad (7)$$

(7) is a **relation between the angles and two sides.**

From (6),

$$\cos a \sin b = \sin a \cos b \cos C + \sin c \cos A;$$

$$\therefore \frac{\cos a}{\sin a} \sin b = \cos b \cos C + \frac{\sin c}{\sin a} \cos A;$$

$$\therefore \cot a \sin b = \cos b \cos C + \sin C \cdot \frac{\cos A}{\sin A}; \qquad \text{(Art. 180.)}$$

$$\cot a \sin b = \cos b \cos C + \sin C \cot A. \qquad (8)$$

The student may derive the five corresponding formulæ:

184. *To express* $\cos \frac{1}{2} A$, $\sin \frac{1}{2} A$, $\tan \frac{1}{2} A$ *in terms of the sides.*

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\cos A + 1}{2}}. \qquad \text{(Art. 92.)}$$

From (2), $$\sqrt{\frac{\cos A + 1}{2}} = \sqrt{\frac{\sin b \sin c + \cos a - \cos b \cos c}{2 \sin b \sin c}}$$

$$= \sqrt{\frac{\cos a - \cos(b + c)}{2 \sin b \sin c}} \qquad \text{(Art. 75.)}$$

$$= \sqrt{\frac{\sin\left(\frac{a+b+c}{2}\right) \sin\left(\frac{b+c-a}{2}\right)}{\sin b \sin c}}. \quad (\alpha) \text{(Art. 82.)}$$

If we put $a + b + c = 2s$, (α) may be written:

$$\left.\begin{aligned}
\cos \tfrac{1}{2} A &= \sqrt{\frac{\sin s \cdot \sin(s - a)}{\sin b \cdot \sin c}};\\
\text{similarly,} \quad \cos \tfrac{1}{2} B &= \sqrt{\frac{\sin s \cdot \sin(s - b)}{\sin a \sin c}},\\
\cos \tfrac{1}{2} C &= \sqrt{\frac{\sin s \cdot \sin(s - c)}{\sin a \sin b}},
\end{aligned}\right\} \qquad (9)$$

Exercise. The student may show that

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}, \tag{10}$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)(s-c)}{\sin s \cdot \sin(s-a)}}, \tag{11}$$

with the corresponding formulæ for B and C.

185. *To find* $\cos \frac{1}{2} a$, $\sin \frac{1}{2} a$, $\tan \frac{1}{2} a$. Using the results of (5), the student may show that, if $S = \frac{1}{2}(A + B + C)$,

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \cdot \sin C}}; \tag{12}$$

$$\sin \tfrac{1}{2} a = \sqrt{\frac{1-\cos a}{2}},$$

$$= \sqrt{\frac{\sin B \sin C - \cos A - \cos B \cos C}{2 \sin B \sin C}},$$

$$= \sqrt{\frac{-(\cos(B+C) + \cos A)}{2 \sin B \sin C}}.$$

$$\therefore \sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}}, \tag{13}$$

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}. \tag{14}$$

$$\sin a = 2 \sin \tfrac{1}{2} a \cos \tfrac{1}{2} a \qquad \text{(Art. 91.)}$$

$$= \frac{2}{\sin B \sin C} \sqrt{-\cos S \cos(S-A) \cos(S-B) \cos(S-C)}. \tag{15}$$

186. I. Since any side $a < 180°$, and any angle $A < 180°$,

$$\cos \tfrac{1}{2} a,\ \sin \tfrac{1}{2} a,\ \tan \tfrac{1}{2} a,\ \sin \tfrac{1}{2} A,\ \cos \tfrac{1}{2} A,\ \tan \tfrac{1}{2} A$$

are all positive; hence the sign of the radical in expressions (9)–(15) is positive.

II. Since $s < 180°$, $a < 180°$, $b < 180°$, $c < 180°$, and since the differences $s - a$, $s - b$, $s - c$ are less than 180° and positive, the expressions (9) – (11) are real.

III. If a', b', c' are the sides of the polar triangle of A, B, C, then $a' < (b' + c')$;

i.e. $$180° - A < (180° - B + 180° - C);$$

i.e. $$B + C - A < 180°;$$

i.e. $$S - A < 90°;$$

$$\therefore \cos(S - A) \text{ is positive.}$$

Since $270° > S° > 180°$, $\cos S$ is negative; *i.e.* $-\cos S$ is positive. Therefore the expressions (12) – (15) are real.

187. $\cos \frac{1}{2}(A+B) = \cos \frac{1}{2}A \cos \frac{1}{2}B - \sin \frac{1}{2}A \sin \frac{1}{2}B$. (Art. 75.)

Substitute in this equation the values of $\cos \frac{1}{2}A$, etc., from (9) and (10).

$$\cos \tfrac{1}{2}(A+B) = \frac{\sin s}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin b \sin a}}$$

$$-\frac{\sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin b \sin a}}.$$

$$\therefore \cos \tfrac{1}{2}(A+B) = \frac{\sin s - \sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)(s-c)}{\sin b \sin a}}.$$

Now $$\frac{\sin s - \sin(s-c)}{\sin c} = \frac{2\cos\frac{2s-c}{2}\cdot\sin\frac{c}{2}}{2\sin\frac{1}{2}c\cos\frac{1}{2}c} \qquad \text{(Art. 83.)}$$

$$= \frac{\cos\frac{a+b}{2}}{\cos\frac{1}{2}c}. \qquad \text{(Art. 91.)}$$

And $$\sin\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}. \quad \text{(from (10))}$$

$$\therefore \cos\frac{A+B}{2} = \frac{\cos\frac{a+b}{2}}{\cos\frac{c}{2}}\sin\frac{C}{2}. \qquad (16)$$

Exercise. The student may prove:

$$\cos\frac{A-B}{2} = \frac{\sin\frac{a+b}{2}}{\sin\frac{c}{2}}\sin\frac{C}{2}; \qquad (17)$$

$$\sin\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}\cos\frac{C}{2}; \qquad (18)$$

$$\sin\frac{A-B}{2} = \frac{\sin\frac{(a-b)}{2}}{\sin\frac{C}{2}}\cos\frac{C}{2}. \qquad (19)$$

Formulæ (16), (17), (18), (19) are known as **Gauss' Formulæ.**

188. The student will prove without difficulty:

$$\tan\frac{A+B}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2}; \tag{20}$$

$$\tan\frac{A-B}{2}=\frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2}; \tag{21}$$

$$\tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}; \tag{22}$$

$$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2}. \tag{23}$$

Formulæ (20), (21), (22), (23) are known as **Napier's Analogies.**

189. In the special case that one of the angles, C, is equal 90°, we derive from the foregoing formulæ the following:

From (1), $$\cos c=\cos a\cos b. \tag{24}$$

From (3), $$\left.\begin{aligned}\sin A&=\frac{\sin a}{\sin c};\\ \sin B&=\frac{\sin b}{\sin c}.\end{aligned}\right\} \tag{25}$$

From (7), $$\left.\begin{aligned}\cos A&=\cos a\sin B;\\ \cos B&=\cos b\sin A.\end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned}\therefore\ \cos A&=\frac{\tan b}{\tan c};\\ \cos B&=\frac{\tan a}{\tan c}.\end{aligned}\right\} \tag{27}$$

From (8), $$\left.\begin{aligned}\tan A&=\frac{\tan a}{\sin b};\\ \tan B&=\frac{\tan b}{\sin a}.\end{aligned}\right\} \tag{28}$$

From (5),

$$\left.\begin{aligned} \cos a &= \frac{\cos A}{\sin B}; \\ \cos b &= \frac{\cos B}{\sin A}; \\ \therefore\ \cos c &= \frac{\cos A}{\sin A} \cdot \frac{\cos B}{\sin B} = \cot A \cdot \cot B. \end{aligned}\right\} \quad (29)$$

190. Definition. Two angles are said to be of the "same affection," if both are greater or both are less than 90°. They are said to be of **opposite affection** if one is greater and the other less than 90°.

Formula (24) shows that if a and b are of the *same affection*, $c < 90°$; and if *a and b are of opposite affection*, $c > 90°$, C being a right angle. For, if a and b are both of the same affection, $\cos a \cdot \cos b$ is positive, $\therefore \cos c$ is positive, $\therefore c < 90°$. But if they are of opposite affection, $\cos a \cdot \cos b$ is negative, $\therefore \cos c$ is negative; $\therefore c > 90°$.

Exercise 1. Show that in a right-angled triangle an angle and the side opposite are of the same affection.

Exercise 2. Show that a side and the hypotenuse of a right-angled triangle are of the same or opposite affection, according as the included angle is less than or greater than 90°.

Exercise 3. In *any* triangle, $\frac{a+b}{2}$ and $\frac{A+B}{2}$ are of the same affection.

Exercise 4. If C is a right angle, show that

$$2 \cos c = \cos (a + b) + \cos (a - b).$$

Exercise 5. If C is a right angle, show that

$$\tan \tfrac{1}{2} (c + a) \tan (c - a) = \tan^2 \tfrac{1}{2} b.$$

CHAPTER XIX

SOLUTION OF SPHERICAL TRIANGLES

191. According to propositions proved in Geometry we are able to construct a spherical triangle when *any three* of its parts are given. Accordingly, we propose the problem, to show that if **given any three parts of a spherical triangle, we are able to calculate** the remaining parts.

This problem is known as the solution of spherical triangles.

There are six cases; viz. when there are given:

I. Three sides.
II. Two sides and included angle.
III. Two sides and the angle opposite one of them.
IV. Two angles and a side opposite one of them.
V. One side and two angles adjacent to it.
VI. Three angles.

192. The Right Triangle.

Before considering the solution of the general triangle, we shall discuss the solution of the right triangle. Let C be a right angle.

We have the following cases:

Given:		To calculate:
I.	$a, b.$	$c, A, B.$
II.	$a, c.$	$b, A, B.$
	$c, b.$	$a, A, B.$
III.	$a, A.$	$b, c, B.$
	$b, B.$	$a, c, A.$
IV.	$a, B.$	$b, c, A.$
	$b, A.$	$a, c, B.$
V.	$c, A.$	$a, b, B.$
	$c, B.$	$a, b, A.$
VI.	$A, B.$	$a, b, c.$

193. If any of the [illegible] its sine, then the solution [illegible] two angles less than 180° [illegible] below.

If, however, all the [illegible] their sines, or [illegible]

194. Discussion of the [illegible] Art. 189, the student will [illegible] instances he will find [illegible] problems.

Case I.

Given a, b; required c, A, B.

$$\sin c = \frac{\sin a \sin b}{\sin A},$$

$$\tan A = \frac{\sin a}{\sin b},$$

$$\tan B = \frac{\sin b}{\sin a}.$$

The solution is unique.

Case II.

Given a, c; required b, A, B.

$$\sin b = \frac{\sin a \sin c}{\sin A},$$

$$\sin A = \frac{\sin a \sin c}{\sin b},$$

$$\sin B = \frac{\sin c \sin a}{\sin b}.$$

This case presents no apparent ambiguity. But by Art. 189, [illegible] two of the same affection, [illegible] the solution is unique.

Case III.

Given a, b; required c, A, B.

$$\sin c = \frac{\sin a}{\sin A},$$

$$\sin b = \frac{\tan a}{\tan A},$$

$$\sin B = \frac{\cos A}{\cos a}.$$

[illegible] are given by their sines; hence there is, apparently [illegible] of ambiguities.

(*a*) If $\sin a = \sin A$, then $a = A$. (Art. 190, Exercise 1.)

$$\therefore \sin c = 1, \sin b = 1, \sin B = 1.$$

Hence in this case the solution is unique.

(*b*) There remain yet two possibilities for c; viz.:

I. c and a of like affection; then $b < 90°$, and $B < 90°$.
(Art. 190, Exercises 1 and 2.)

II. c and a of opposite affection; then $b > 90°$ and $B > 90°$.
These are the only alternatives.

Case IV.

Given a, B; required c, b, A.

$$\tan c = \frac{\tan a}{\cos B};$$

$$\cos A = \cos a \sin B;$$

$$\tan b = \sin a \tan B.$$

The solution is unique. Why?

Case V.

Given c, A; required a, b, B.

$$\sin a = \sin c \sin A;$$

$$\tan b = \tan c \cos A;$$

$$\cot B = \cos c \tan A.$$

The solution is unique. Why?

Case VI.

Given A, B; required a, b, c.

$$\cos a = \frac{\cos A}{\sin B};$$

$$\cos b = \frac{\cos B}{\sin A};$$

$$\cos c = \cot A \cot B.$$

195. Solution of the general triangle.

Case I. (Art. 194.)

Given a, b, c; required A, B, C.

This may be solved by (2),

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

This formula, however, is not adapted to logarithmic computation. When logarithms are employed, either (9), (10), or (11) may be used. If, however, *all* the angles are desired, (11) serves us best.

The solution is unique. Why?

EXAMPLE. Given

$$a = 93° 45',$$
$$b = 27° 16',$$
$$c = 88° 12'.$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \cdot \sin (s-a)}}.$$

$$\therefore \log \tan \tfrac{1}{2} A = \tfrac{1}{2} [\log \sin (s-b) + \log \sin (s-c) - \log s - \log (s-a)].$$

$$s = 104° 36' 30'',$$
$$s - a = 10° 51' 30'',$$
$$s - b = 77° 20' 30'',$$
$$s - c = 16° 24' 30''.$$

$$\log \sin (s-b) + \log \sin (s-c) = 9.98936 + 9.45099 = 9.44035;$$
$$\log \sin s + \log \sin (s-a) = 9.98572 + 9.27504 = 9.26016;$$
$$\therefore \log \tan \tfrac{1}{2} A = .09009;$$
$$\therefore A = 101° 48' 4''.$$

Case II.

Given a, b, C; required A, B, c.

Formulæ (20) and (21) give

$$\tan \frac{(A+B)}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \cot \frac{C}{2};$$

$$\tan \frac{(A-B)}{2} = \frac{\sin \frac{(a-b)}{2}}{\sin \frac{(a+b)}{2}} \cot \frac{C}{2}.$$

Whence $A + B$ and $(A - B)$ can be found, and therefore A and B.

From (16), $$\cos \frac{c}{2} = \frac{\cos \frac{a+b}{2}}{\cos \frac{A+B}{2}} \sin \frac{C}{2},$$

which determines c.

If we wish to determine c independent of A and B, proceed as follows:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C = \cos a (\cos b + \tan a \sin b \cos C).$$

Put $\tan a \cos C = \tan \phi$, which determines ϕ.

$$\therefore \cos c = \cos a \frac{(\cos b \cos \phi + \sin b \sin \phi)}{\cos \phi};$$

$$\therefore \cos c = \frac{\cos a \cos (b - \phi)}{\cos \phi}.$$

This formula is adapted to logarithmic computation.

Case III.

Given a, b, A; required B, c, C.

$$\sin B = \frac{\sin A}{\sin a} \sin b \ (\alpha); \qquad \text{(from (3))}$$

$$\tan \frac{c}{2} = \frac{\cos \frac{A+B}{2}}{\cos \frac{A-B}{2}} \tan \frac{a+b}{2}; \qquad \text{(from (22))}$$

$$\cot \frac{C}{2} = \frac{\cos \frac{a+b}{2}}{\cos \frac{(a-b)}{2}} \tan \frac{(A+B)}{2}. \qquad \text{(from (20))}$$

Discussion of the solution.

I. If $\sin A \sin b > \sin a$, no solution is possible; for $\sin B > 1$.

II. If $\sin a > \sin A \sin b$, equation (α) is satisfied by two supplementary values of B. Now $\frac{A+B}{2}$ and $\frac{a+b}{2}$ are of like affection, hence, if both values of B satisfy this condition, there are two solutions; otherwise, but one. This test requires that we first solve for B before we determine whether there are one or two solutions. There are several methods of determining this by an inspection of given data. The following is one, quoted from Wheeler's Trigonometry, where the reader is referred to Chauvenet's Trigonometry, p. 197:

(1) When b differs more from 90° than a, B must be in the same quadrant as b (Art. 177), and there can be but one solution. It remains to show that (2) when a differs more from 90° than b, there will be necessarily two solutions. We have

$$\sin c = \frac{\cos a - \cos b \cos c}{\sin b \cos A} \qquad \text{(from (1))}$$

Two solutions exist so long as $\sin c$ is positive. Now, when a differs more from 90° than b, we have (neglecting the signs for a moment),

$$\cos a > \cos b > \cos b \cos c;$$

therefore, the numerator of the above value of $\sin c$ has the sign of $\cos a$. But by Art. 177, a and A are in the same quadrant, and therefore $\cos a$ and $\cos A$ have the same sign; therefore, the numerator and denominator have the same sign, and the value of $\sin c$ is positive, as was to be proved.

Hence, there is but one solution when the side opposite the given angle differs less from 90° than the other given side, and two solutions when the side opposite the given angle differs more from 90° than the other given side.

Case IV.

Given a, A, B; required C, b, c.

$$\sin b = \frac{\sin B}{\sin A} \cdot \sin a;$$

$$\tan \frac{c}{2} = \frac{\cos \frac{A+B}{2}}{\cos \frac{A-B}{2}} \tan \frac{a+b}{2};$$

$$\tan \frac{C}{2} = \frac{\sin \frac{(a-b)}{2}}{\cos \frac{a+b}{2}} \cot \frac{A+B}{2}.$$

This solution is also ambiguous.

If $\sin a \sin B > \sin A$, no solution is possible.

Now it is evident that the triangle ABC, and its polar triangle $A'B'C'$, have the same number of solutions.

The data given in this case determine *two sides* of $A'B'C'$ and *one angle* of $A'B'C'$ opposite one of these sides. Hence, by Case III., we can determine whether there are one or two solutions of $A'B'C'$; and, hence, whether there are one or two triangles of ABC satisfying the given data.

Case V.

Given c, A, B; required C, a, b.

$$\tan \frac{a+b}{2} = \frac{\cos \frac{A-B}{2}}{\cos \frac{A+B}{2}} \tan \frac{c}{2}; \qquad \text{(from (22))}$$

$$\tan \frac{a-b}{2} = \frac{\sin \frac{A-B}{2}}{\sin \frac{A+B}{2}} \tan \frac{c}{2}.$$

This determines $\frac{a+b}{2}$ and $\frac{a-b}{2}$, and hence a and b.

$$\tan\frac{C}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{A+B}{2}. \qquad \text{(from (20))}$$

The solution is unique.

Case VI.

Given A, B, C; required a, b, c.

$$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}; \qquad \text{(from (12))}$$

or, if all the sides are desired,

$$\tan\tfrac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}. \qquad \text{(from (14))}$$

Formula (3) may be used as a check formula in each of the cases.

EXAMPLES. XLIII.

1. $a=84^\circ\,30'$; $b=46^\circ\,10'$; $C=75^\circ\,46'$. Find A, B, c.
2. In a right-angled triangle calculate A, B, c, having given
$$a=86^\circ\,45'; \quad b=108^\circ\,20'.$$
3. In a right-angled triangle,
$$a=63^\circ\,12'; \quad B=75^\circ\,9'. \quad \text{Find } A,\ B,\ c.$$
4. A spherical triangle has,
$$a=124^\circ\,12'\,30''; \quad b=54^\circ\,18'; \quad c=97^\circ\,12'\,30''. \quad \text{Solve the triangle.}$$
5. In a right triangle,
$$A=125^\circ; \quad B=43^\circ\,18'. \quad \text{Solve the triangle.}$$
6. A spherical triangle has,
$$a=125^\circ\,12'; \quad b=107^\circ\,10'; \quad c=45^\circ. \quad \text{Solve the triangle.}$$
7. Having given,
$$b=108^\circ\,30'; \quad c=40^\circ\,50'; \quad C=39^\circ\,50'. \quad \text{Solve the triangle.}$$
8. A spherical triangle has,
$$a=110^\circ\,30'; \quad c=16^\circ\,12'; \quad C=84^\circ\,2'. \quad \text{Solve the triangle.}$$
9. Having given,
$$A=66^\circ\,30'; \quad B=75^\circ\,20'; \quad C=110^\circ\,10'. \quad \text{Solve the triangle.}$$
10. In a spherical triangle,
$$A=107^\circ\,36'\,9''; \quad B=36^\circ\,17'\,20''; \quad c=56^\circ\,21'\,30''. \quad \text{Solve the triangle.}$$
11. In a right triangle,
$$a=75^\circ\,24'; \quad c=69^\circ\,20'. \quad \text{Solve the triangle.}$$

12. In a spherical triangle,

$$a = 147^\circ\ 49'\ ;\ A = 137^\circ\ 36'\ ;\ C = 90^\circ.\quad \text{Find } b,\ B,\ c.$$

13. In a right triangle,

$$A = 73^\circ\ 7'\ ;\ c = 114^\circ\ 32'\ ;\ C = 90^\circ.\quad \text{Find } a,\ B,\ b.$$

14. A spherical triangle,

$$C = 146^\circ\ 37'\ 54''\ ;\ B = 55^\circ\ 36''\ ;\ b = 96^\circ\ 34'\ 24''.\quad \text{Solve the triangle.}$$

15. In a right triangle,

$$A = 85^\circ\ 25'\ ;\ B = 105^\circ\ 12';\ C = 90^\circ.\quad \text{Solve the triangle.}$$

16. In a spherical triangle,

$$A = 39^\circ\ 45'\ ;\ B = 26^\circ\ 59'\ ;\ c = 154^\circ\ 47'.\quad \text{Solve the triangle.}$$

17. A spherical triangle has,

$$a = 150^\circ\ 57'\ ;\ b = 134^\circ\ 16'\ ;\ A = 144^\circ\ 23'.\quad \text{Solve the triangle.}$$

18. In a spherical triangle,

$$a = 35^\circ\ 37'\ ;\ b = 59^\circ\ 12'\ ;\ C = 124^\circ\ 18'.\quad \text{Solve the triangle.}$$

19. In a spherical triangle,

$$A = 58^\circ\ 16'\ ;\ B = 41^\circ\ 28'\ ;\ c = 75^\circ\ 54'.\quad \text{Solve the triangle.}$$

20. In a spherical triangle,

$$a = 68^\circ\ 16'\ ;\ b = 53^\circ\ 24'\ ;\ C = 86^\circ\ 40'.\quad \text{Solve the triangle.}$$

21. In a right triangle,

$$A = 29^\circ\ 33'\ ;\ B = 87^\circ\ 21'\ ;\ C = 90^\circ.\quad \text{Solve the triangle.}$$

ANSWERS TO EXAMPLES

II.

1. 80. 2. 10. 3. $109\frac{11}{21}$. 4. 5 acres. 5. $\frac{1760\,a}{b}$. 6. $\frac{a \cdot c}{3}$ yds.

III.

(i.) 1. .09175 of a right angle.
2. .0675 of a right angle.
3. 1.07875 of a right angle.
4. .180429 of a right angle.
5. $1.46\dot{7}$ of a right angle.
6. $.5\dot{4}$ of a right angle.
7. 1° 14′ 15″.
8. 7° 52′ 30″.
9. 153° 24′ 29.34″.
10. 21° 36′ 8.1″.
11. 16° 12′ 37.26″.
12. 31° 30′.

(ii.) 1. 2 right angles, or 180°.
2. $\frac{3}{2}$ of a right angle.
3. $\frac{2}{\pi}$ right angles.
4. $\frac{6}{\pi}$ right angles.
5. 2 right angles.
6. $\frac{4}{\pi^2}$ right angles.
7. $\frac{2\,\theta}{\pi}$ right angles.
8. .002 of a right angle.
9. 20 right angles.

(iii.) 1. π. 2. 2π. 3. $\frac{\pi}{3}$. 4. $\frac{\pi}{8}$. 5. $\frac{\pi}{180}$. 6. 1^c. 7. $\frac{n}{180}\pi$. 8. $\frac{1}{2}^c$. 9. $\frac{A\pi}{180}$.

(v.) 1. $\frac{1}{3}$. 2. $\frac{\pi}{180}$. 3. $\frac{43}{20}$.

IV.

1. $\frac{3}{2}$. 2. 90. 3. $4\frac{4}{5}$. 4. $112\frac{1}{2}$ ft.
5. $5\frac{37}{63}$ ft. 6. 838000 miles.
7. $\frac{1}{9}$ radian $= 6\frac{4}{11}$ degrees.
8. $21\frac{21}{44}$ degrees. 9. $51\frac{6}{11}''$.
10. about 34 yds. 11. 1 : 3.1416.
12. 3.1416. 13. 3.1416.
14. 400 : 1. 15. .0000484 ....
16. $49\frac{1}{11}$ in. 18. 473 : 489.
19. (i.) $k = 1$. (ii.) $k = \frac{180}{\pi}$.
20. 38°, 18°. 21. $\frac{n\pi}{10800}$.
22. (i.) 120°, $\frac{2\pi}{3}$; (ii.) 135°, $\frac{3\pi}{4}$.
(iii.) 156°, $\frac{13}{15}\pi$.
23. (i.) $3\frac{3}{4}$; (ii.) $\frac{15}{2\pi}$. 24. $\frac{19°}{10}$.
25. $\frac{9\,a + 10\,b}{10\,c}$. 26. $\frac{1800\,\pi}{19\,\pi + 1800}$.
27. 9 or 16.

V.

[illegible] $\tan A = \frac{3\sqrt{41}}{25}$. 2. $\frac{1}{\sqrt{5}}$; $\frac{2}{\sqrt{5}}$; 2.

[illegible] (ii.) $\frac{CD}{CB}, \frac{CB}{CA}$; (iii.) $\frac{DB}{CD}, \frac{BA}{CB}$; (iv.) $\frac{DB}{AB}, \frac{BC}{AC}$;

(v.) $\frac{AD}{AB}, \frac{AB}{AC}$; (vi.) $\frac{DB}{AD}, \frac{BC}{AB}$.

[illegible] (ii.) $\frac{BA}{EA}$ or $\frac{AC}{EC}$; (iii.) $\frac{DC}{BC}$; (iv.) $\frac{AB}{AE}$; (v.) $\frac{AD}{AB}$ or $\frac{AB}{AC}$;

[illegible] (vii.) $\frac{DB}{CD}$ or $\frac{BA}{CB}$ or $\frac{AE}{CA}$; (viii.) $\frac{DA}{BD}$; (ix.) $\frac{BA}{EB}$ or $\frac{AC}{EA}$;

(x.) $\frac{DC}{BD}$; (xi.) $\frac{DB}{AB}$ or $\frac{BC}{AC}$; (xii.) $\frac{BE}{AE}$.

[illegible] the smaller angle, the sine $= \frac{5}{13}$, cosine $= \frac{12}{13}$, tangent $= \frac{5}{12}$.
[illegible] the larger angle, the sine $= \frac{12}{13}$, cosine $= \frac{5}{13}$, tangent $= \frac{12}{5}$.

[illegible] the smaller angle, the sine $= \frac{1}{2}$, cosine $= \frac{\sqrt{3}}{2}$, tangent $= \frac{1}{\sqrt{3}}$.

[illegible] the larger angle, the sine $= \frac{1}{2}\sqrt{3}$, cosine $= \frac{1}{2}$, tangent $= \sqrt{3}$.

[illegible] $\sqrt{3}$; $\sin A = \frac{1}{2}\sqrt{3}$; $\cos A = \frac{1}{2}$; $\tan A = \sqrt{3}$.

[illegible] $\sqrt{2}$; $\sin B = \sqrt{\frac{2}{3}}$; $\sin A = \frac{1}{\sqrt{3}}$.

VII.

[illegible] 79 ft.
[illegible] 46 ft.
[illegible] 6 ft.
[illegible] 5 ft.
[illegible] ft.
[illegible] 60°; 173 ft.

7. 63.17 yds.
8. 192.8 ft.
9. 34.15 ft.
10. 73.2 ft.
11. 86.6 ft.
12. 1524 yds.
13. 173.2 yds.
14. 373 ft.
15. $\frac{\sqrt{3}\,a}{3\,b}$.
16. 30°.
17. About 523.6 miles.

XI.

1. [illegible]
2. [illegible] 100°.
3. 0°.
4. $-260°$.
5. 115°.
6. 410°.
7. $-\frac{1}{4}\pi$.
8. $\frac{2}{3}\pi$.

XII.

1. [illegible]; $-\frac{\sqrt{3}}{2}$; $-\frac{1}{\sqrt{3}}$.
2. [illegible]; $-\frac{1}{\sqrt{2}}$; -1.
3. [illegible] $\frac{\sqrt{3}}{2}$; $-\frac{1}{2}$; $-\sqrt{3}$.
4. [illegible]; $+\frac{\sqrt{3}}{2}$; $-\frac{1}{\sqrt{3}}$.
5. $-\frac{1}{\sqrt{2}}$; $\frac{1}{\sqrt{2}}$; -1.
6. $\frac{\sqrt{3}}{2}$; $+\frac{1}{2}$; $+\sqrt{3}$.
7. $-\frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}}$; $+1$.
8. $-\frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}}$; $+1$.
9. $+\frac{1}{2}$; $+\frac{\sqrt{3}}{2}$; $+\frac{1}{\sqrt{3}}$.
10. $+\frac{1}{2}$; $+\frac{\sqrt{3}}{2}$; $+\frac{1}{\sqrt{3}}$.
11. $-\frac{\sqrt{3}}{2}$; $-\frac{1}{2}$; $+\sqrt{3}$.
12. $-\frac{\sqrt{3}}{2}$; $+\frac{1}{2}$; $-\sqrt{3}$.
13. $+\frac{1}{\sqrt{2}}$; $+\frac{1}{\sqrt{2}}$; $+1$.
14. $\frac{\sqrt{3}}{2}$; $-\frac{1}{2}$; $-\sqrt{3}$.
15. $-\frac{1}{2}$; $-\frac{\sqrt{3}}{2}$; $+\frac{1}{\sqrt{3}}$.

16. 36°; 150°; −210°; −330°.
17. 45°; 135°; −225°; −315°.
18. 60°; 120°; −240°; −300°.
19. −30°; −150°; 210°; 330°.
20. 20°; 160°; 380°; 520°.
21. $\frac{5}{4}\pi$; $\frac{7}{4}\pi$; $\frac{13}{4}\pi$; $\frac{15}{4}\pi$.
22. $\frac{5}{7}\pi$; $\frac{13}{7}\pi$; $\frac{23}{7}\pi$; $\frac{27}{7}\pi$.
25. The tangent. 26. No.

XV. (a).

1. $$\sin A = \pm\sqrt{1-\cos^2 A};\quad \tan A = \pm\frac{\sqrt{1-\cos^2 A}}{\cos A};$$
$$\cot A = \pm\frac{\cos A}{\sqrt{1-\cos^2 A}};\quad \sec A = \frac{1}{\cos A};\quad \csc A = \pm\frac{1}{\sqrt{1-\cos^2 A}}.$$

2. $$\sin(90^\circ + A) = \frac{\cot A}{\sqrt{\cot^2 A + 1}};\quad \cos(90^\circ + A) - \frac{1}{\sqrt{\cot^2 A + 1}};$$
$$\tan(90^\circ + A) = -\cot A;\quad \sec(90^\circ + A) = -\sqrt{\cot A + 1};$$
$$\csc(90^\circ + A) = \frac{\sqrt{\cot^2 A + 1}}{\cot A}.$$

3. $\sin(180^\circ - A) = \pm\dfrac{\sqrt{\sec^2 A - 1}}{\sec A}$; $\cos(180^\circ - A) = -\dfrac{1}{\sec A}$;
$\tan(180^\circ - A) = \pm\sqrt{\sec A - 1}$, etc.

4. $\sin A = \dfrac{1}{\csc A}$; $\cos A = \dfrac{\pm\sqrt{\csc^2 A - 1}}{\csc A}$; $\tan A = \dfrac{1}{\pm\sqrt{\csc^2 A - 1}}$, etc.

5. $\sin A = \sin A$; $\cos A = \pm\sqrt{1 - \sin^2 A}$; $\tan A = \pm\dfrac{\sin A}{\sqrt{1 - \sin^2 A}}$, etc.

6. See answers to Example 1.

(b) 1. $\tan A = \pm\frac{3}{4}$; $\operatorname{cosec} A = \frac{5}{3}$.

2. $\sin B = \dfrac{2\sqrt{2}}{3}$; $\cot B = \dfrac{1}{2\sqrt{2}}$.

3. $\sin A = \frac{4}{5}$; $\sec A = \frac{5}{3}$.

4. $\cot\theta = \dfrac{1}{\sqrt{15}}$; $\sin\theta = \dfrac{\sqrt{15}}{4}$.

5. $\sin\theta = \dfrac{\sqrt{3}}{2}$; $\cos\theta = \frac{1}{2}$.

6. $\sin\theta = \dfrac{\sqrt{5}}{3}$; $\sin\theta = \frac{2}{3}$.

7. $\dfrac{b}{\pm\sqrt{c^2 - b^2}}$.

8. $\dfrac{a}{\pm\sqrt{a^2+1}}$; $\dfrac{1}{\sqrt{a^2+1}}$.

9. $\sin\theta = \dfrac{\sqrt{a^2-1}}{a}$, $\cot\theta = \dfrac{1}{\sqrt{a^2-1}}$.

11. $h^2(1 + k^2) = 1$.

12. $\cos\theta = \frac{4}{5}$; $\tan\theta = \frac{3}{4}$; $\cot\theta = \frac{4}{3}$; $\sec\theta = \frac{5}{4}$; $\csc\theta = \frac{5}{3}$.

XVI.

1. 45°; 135°. 2. 30°; 150°; 210°; 330°.
3. 45°; 135°; 225°; 315°.
4. 60°; 120°; 240°; 300°.
5. 30°; 150°; 210°; 330°.
6. 30°; 150°; 210°; 330°.
7. 30°; 150°.
8. 45°; 315°; 0°; 180°.
9. 90°; 270°; 60°; 120°.
10. 60°; 120°; 240°; 300°.
11. 45°; 225°.
12. 90°; 70°; 45°; 315°.
13. 0°; 180°; 45°; 135°.

14. 45°; 135°; 225°; 315°. 15. 45°; 135°. 16. 30°; 330°. 17. 30°; 150°.
18. ± 60°; ± 120°; ± 240°; ± 300°.
19. 0°; 180°; − 180°; − 360°.
20. $2n\pi \pm \frac{\pi}{4}$; (n any integer).
21. $n\pi$; $2n\pi + \frac{\pi}{3}$; $(2n+1)\pi - \frac{\pi}{3}$.
22. $n\pi + \frac{\pi}{4}$.

XXI.

1. $\sin(\theta + \phi) + \sin(\theta - \phi)$.
2. $\cos(\alpha - \beta) + \cos(\alpha + \beta)$.
3. $\sin(2\alpha + 3\beta) + \sin(2\alpha - 3\beta)$.
4. $\cos 2\alpha + \cos 2\beta$.
5. $\sin 8\theta - \sin 2\theta$. 6. $\cos\theta + \cos 2\theta$. 7. $\frac{1}{2}(\cos 3\theta - \cos 5\theta)$.
8. $\frac{1}{2}(\sin 4\theta - \sin\theta)$.
9. $\sin 60° + \sin 40°$.
10. $\frac{1}{2}(\sin 60° - \sin 30°)$.
11. $2\cos 3\theta \cos 2\theta$.
12. $-\cos 4\theta \sin 2\theta$.
13. $4\cos^2\frac{\theta}{2}\sin 2\theta$.

XXV.

1. $\frac{3}{\sqrt{10}}$. 2. $\frac{2}{\sqrt{3}}$. 3. $\frac{2}{\sqrt{3}}$, $-\sqrt{\frac{2}{3}}$. 13. 1.

XXVI.

1. 60°. 2. 120°. 3. 30°; 150°. 4. 135°. 5. 45°; 135°. 6. 120°.

XXVIII.

1. $\cos A = \frac{1}{2}$; $\cos\frac{1}{2}A = \frac{1}{2}\sqrt{3}$.
2. 45°; 60°; 75°.
3. 135°; 30°; 15°. 4. 3. 5. 14.
6. $1 + \sqrt{3}$. 7. 120°. 8. 120°.
9. 120°. 10. 90°; 36° 52′.
11. 130° 27′. 12. 125° 6′.
13. 120°. 14. $a = 1$.
15. $C = 30°$; $a = \sqrt{3} + 1$; $b = 2$.
16. $A = 75°$; $a = b = \sqrt{3} + 1$.
17. $C = 60°$ or 120°.
18. 60°; 75°; 6 yds.
21. $15 : 8\sqrt{3} : 4\sqrt{5} + 6$.

XXIX.

1. 3; $\frac{10}{3}$; $\frac{1}{4}$; $\frac{3}{2}$; $-\frac{5}{2}$. 3. 2; 4; −1; −3; −4. 5. $\frac{4}{3}$; $\frac{2}{3}$; $\frac{1}{3}$; $\frac{4}{3}$; $\frac{1}{3}$.
2. 3; 6; −1; −3; −6, 2. 4. 3; −1; 5; −2; 3; −3.

XXX.

1. 3.55169; .87251; $\bar{3}.63548$. 3. $\bar{2}.65920$; $\bar{2}.54926$; $\bar{3}.85941$.
2. 4.93662; 1.98320; $\bar{3}.54654$. 4. 6.83181; 7.51129.

XXXI.

1. 3.7040. 2. 45740.2. 3. 2492830. 4. .00043965. 5. 5.68915.

XXXII.

1. 7.669. 4. 55460. 7. 8287. 10. .6731. 13. 2.624.
2. 3.809. 5. 12.03. 8. 1165. 11. 1.096. 14. 22.51.
3. 47.32. 6. .04023. 9. .3107. 12. 823.6. 15. 23.28.

XXXIII.

1. $C = 60.42$; $b = 59.1$; $B = 78°$. 2. $b = 21.14$; $A = 52° 49' 55''$; $a = 27.89$.

XXXIV.

1. $A = 41° 16' 52''$.
2. $73° 32' 12''$; $62° 46' 18''$.
3. $60° 50' 51''$; $73° 1' 50''$.
4. $47° 25' 40''$.
5. $64° 31' 58''$.
6. $135°$; $30°$; $15°$.

XXXV.

1. $a = 313.46$ yds. 2. 1192.55 yds. 3. 22.415 ft. 4. $a = .01116$; $b = .006962$.

XXXVI.

1. $57° 27' 25''$; $62° 32' 35''$.
2. $64° 26' 47''$; $37° 7' 13'$.
3. $A = 24° 41' 48''$; $c = .5886$.
4. $72° 12' 59''$.
5. 14.

XXXVII.

2. $A = 51° 18' 21''$; $C = 88° 41' 39''$; or $A = 128° 41' 39''$; $C = 11° 18' 21''$.
3. $B = 38° 38' 24''$; $C = 91° 91' 36''$; $c = 155.3$. 4. No.

XXXVIII.

1. $28° 35' 39''$.
2. $104° 44' 39''$.
3. $32° 20' 48''$.
4. $43° 40'$.
5. $128° 23' 13''$.
6. 106531 ft.
7. 3437.6 yds.
8. 1728.2 chains.
9. 25376 yds.
10. $A = 66° 27' 48''$; $B = 12° 55' 12''$.
11. $A = 92° 12' 53''$; $B = 35° 37' 7''$.
12. $B = 29° 1' 40''$; $C = 74° 55' 50''$.
13. $B = 70° 35' 24''$, or $109° 24' 36''$.
14. $B = 51° 56' 17''$, or $128° 3' 43''$.
15. $B = 62° 6' 10''$, or $117° 53' 50''$.
16. Very nearly $90°$.
17. 1319.6 yds.

XLII.

1. $\sin A = \frac{3}{5}$; $\cos A = \frac{4}{5}$. 2. $\frac{10}{3}\sqrt{3}$ ft. $= 46.19\cdots$ ft. 6. 4227.47 ft.

9. $30°, 60°, 90°, 120°$, etc., have for sine $\frac{1}{2}, \sqrt{\frac{3}{2}}, 1, \sqrt{\frac{3}{2}}, \frac{1}{2}, 0, -\frac{1}{2}, -\sqrt{\frac{3}{2}} - 1, -\sqrt{\frac{3}{2}}, -\frac{1}{2}$, respectively.

12. The other sides are 765.4321 ft.; 1006.6 ft.

15. $30°, 60°, 90°$, etc., have for $\tan \frac{1}{3}\sqrt{3}, \sqrt{3}, \infty, -\sqrt{3}, -\frac{1}{3}\sqrt{3}, 0, \frac{1}{3}\sqrt{3}, \infty, -\sqrt{3}, -\frac{1}{3}\sqrt{3}$, respectively.

17. $\frac{168}{193}, \frac{168}{195}, \frac{32592}{193 \times 195}$. 19. $\sec A = \frac{3}{4}\sqrt{2}$. 20. (i.) $\frac{15}{2}$; (ii.) -15.

21. $+\sin 18°, \quad -\cos 18°, \quad -\tan 18°; \quad -\sin 30°, \quad -\cos 36°, \quad +\tan 36°; -\sin 36°, +\cos 36°, -\tan 36°$. 22. $\cos 5a = 16 \cos^5 a - 20 \cos^3 a + 5 \cos a$.

23. (i.) $0, n\pi, \frac{1}{3}\pi$; (ii.) $\cos\theta = \frac{1}{2}$, or $\sin(\theta - 45°) = \frac{1}{\sqrt{2}}$.

24. $\frac{1}{26}\sqrt{651} = .981\cdots$. 25. $\frac{11}{630}A$ radians; $5.72965°$.

26. sine, $\frac{4}{5}$; tan, $\frac{4}{3}$; cot, $\frac{3}{4}$; cosec, $\frac{5}{4}$; sec, $\frac{5}{3}$.

29. (i.) $6\log_{10}3 + 3\log_{10}2 - 3$; (ii.) $\frac{1}{2}\{\log_{10}7 + 1 - \log_{10}2\}$; (iii.) $3\log_{10}7 + 3\log_{10}2 - 2\log_{10}3 - 2$.

31. $\frac{540}{11}\theta$ deg.; 19.09854°.

33. $-\frac{2}{3}\sqrt{2}$.

34. $\frac{2}{3}$; -9.

36. $C = 18°$, $a = c = 2 + \sqrt{(10 - 2\sqrt{5})}$.

39. π; $\frac{1}{3}\pi$; $\frac{5}{3}\pi$.

41. 1 ft., 120°, 30°; or 2 ft., 60°, 90°.

37. $-1320°$.

42. 104° 28′ 39″.

43. $-630°$.

44. $-\frac{1}{3}\sqrt{5}$.

46. 0; π; $\frac{2}{3}\pi$; $\frac{4}{3}\pi$.

48. $\frac{1}{2}\{\sqrt{6} \pm \sqrt{2}\}$ and 15°, 135°; or, 105°, 45°.

49. 9°; 286° 28′ 41.16″; $\frac{7}{9}$.

52. (i.) $n\pi \pm \frac{1}{3}\pi$; (ii.) $\frac{1}{2}n\pi \pm \frac{1}{3}\pi$, or $n\pi + (-1)^n\frac{1}{4}\pi$.

53. $37\frac{1}{2}$ sq. ft.

54. 40° 29′ 19.85″.

55. $\frac{1}{1080}\pi$; $\frac{1}{10}\pi$; $\frac{5}{4}$.

56. $\tan a = 4\sqrt{3}$, $\operatorname{cosec} a = \frac{7}{12}\sqrt{3}$.

58. (i.) $n\pi \pm \frac{1}{4}\pi$; (ii.) $\frac{1}{2}n\pi \pm \frac{1}{4}\pi$; or, $2n\pi \pm \frac{1}{4}\pi$.

59. $2\sqrt{14}$ sq. ft.

60. 38° 25′ 32.7″.

64. (i.) $1 - \log_{10}5 + \frac{1}{2}\log_{10}7$; $1 - 4\log_{10}5 + 5\log_{10}3$; $2 - 5\log_{10}5 - 2\log_{10}3 + 2\log_{10}7$.

66. 192 ft., 185 ft., and 9234 sq. ft.

67. 2.3 radians = 131.779926°.

69. .69897; .77815; $\bar{2}.33445$; $\bar{1}.91480$.

72. 78° 10′; 70° 30′; 9234 sq. ft.

73. $\frac{4}{15}\pi$; $\frac{8}{15}\pi$; $\frac{14}{15}\pi$; $\frac{19}{15}\pi$; $\frac{24}{15}\pi$.

76. 116.6.

78. 135°, 15°; or 45°, 105°.

79. $\frac{2}{21}\pi$; $\frac{4}{21}\pi$; $\frac{6}{21}\pi$; $\frac{8}{21}\pi$; $\frac{10}{21}\pi$; $\frac{12}{21}\pi$.

81. 32.92….

84. 7 ft.; $\sqrt{19}$ ft.; $\frac{15}{4}\sqrt{3}$ sq. ft.

88. 1035.43 ft.; 765.4321 ft.; 66°.

89. 6.981 ft.

90. $\frac{1}{2}$; $-\frac{1}{2}$.

91. 4.49999.

94. 3210.793.

XLIII.

1. $A = 95° 56' 18''$; $B = 46° 7' 15''$; $c = 75° 56' 27''$.

2. $A = 86° 54' 53''$; $B = 108° 18' 23''$; $c = 91° 1' 12''$.

3. $c = 82° 37' 24''$; $A = 64° 9' 34''$; $b = 73° 5' 32''$.

4. $A = 127° 22'$; $B = 51° 18'$; $C = 72° 27'$.

5. $a = 146° 35' 20''$; $b = 27° 19'$; $c = 137° 59' 33''$.

6. $A = 129° 49' 44''$; $C = 46° 32' 48''$; $B = 78° 47' 48''$.

7. $B = 68° 18'$; $A = 132° 33' 48''$; $a = 131° 15' 48''$; or, $B = 111° 42'$; $A = 77° 4' 36''$; $a = 95° 50'$.

9. $a = 69° 56' 30''$; $b = 82° 16' 28''$; $c = 105° 57'$.

10. $a = 72° 44'$; $b = 36° 22'$; $C = 56° 12'$.

11. Impossible.

12. $B = 60° 45'$; $b = 43° 34'$; $c = 52° 11'$; or, $B = 119° 15'$; $b = 136° 25'$; $c = 127° 49'$.

13. $a = 60° 31' 24''$; $B = 143° 50'$; $b = 147° 32' 6''$.

14. $a = 55°$; $c = 138° 10'$; $A = 42° 30'$.

15. $a = 85° 15'$; $b = 105° 15' 15''$; $c = 91° 15'$.

16. $a = 121° 27' 30''$; $b = 37° 15' 30''$; $C = 161° 22'$.

17. $c = 23° 57'$; $B = 59° 11'$; $C = 29° 9'$; or, $c = 55° 42'$; $B = 120° 49'$; $C = 97° 44'$.

18. $c = 82° 16'$;
$A = 29° 2'$;
$B = 45° 44'$.

19. $a = 58° 33' 37''$;
$b = 41° 47' 40''$;
$C = 104° 48' 4''$.

20. $A = 74° 2' 53''$;
$B = 56° 11' 55''$;
$c = 74° 41' 4''$.

21. $a = 29° 26' 30''$;
$b = 84° 37' 15''$;
$c = 85° 19' 2''$.

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